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Analysis III/IV, Solutions 3 (Weeks 5–6)

Starred problems are **more difficult** and are **not for submission**.

Series and continuous functions. Outer measure.

3.1. Prove Proposition 2.14: a series $\sum_{k=1}^{\infty} a_k$ converges if and only if

$\forall \varepsilon > 0\ \exists N \in \mathbb{N}$ such that $\forall n \geq N$ and $\forall m \in \mathbb{N}$ one has $\left|\sum_{k=n}^{n+m} a_k\right| < \varepsilon$.

Solution:

Denote by $s_n = \sum_{k=1}^{n} a_k$ partial sums of the series. Then $\sum_{k=n}^{n+m} a_k = s_{n+m} - s_{n-1}$, so the condition above is equivalent to the sequence s_n being a Cauchy sequence. This is equivalent to convergence of s_n, which is, by definition, equivalent to convergence of the series $\sum_{k=1}^{\infty} a_k$.

3.2. (a) Show that if $\sum_{k=1}^{\infty} a_k$ converges then $\lim_{k\to\infty} a_k = 0$.

(b) Let $a_k \geq 0$ for all $k \in \mathbb{N}$. Show that if $\sum_{k=1}^{\infty} a_k$ converges then $\sum_{k=1}^{\infty} a_k^3$ also converges.

(c) Does the convergence of $\sum_{k=1}^{\infty} a_k$ imply the convergence of $\sum_{k=1}^{\infty} a_k^2$?

(d) ($\star/2$) Is the assertion of (b) true without the assumption of non-negativity of all a_k?

Solution:

(a) Denote by $s_n = \sum_{k=1}^{n} a_k$ partial sums of the series. Then the convergence of $\sum_{k=1}^{\infty} a_k$ implies that the sequences $\{s_n\}$ and $\{s_{n-1}\}$ both converge to some $s \in \mathbb{R}$. Therefore, the sequence of their differences $\{a_n = s_n - s_{n-1}\}$ converges to zero.

(b) According to (a), there exists $N \in \mathbb{N}$ such that for every $n > N$ one has $a_k < 1$. Since $a_k \geq 0$ for all $k \in \mathbb{N}$, we have $0 \leq a_k^3 < a_k$ for $k > N$, which implies the convergence of $\sum_{k=1}^{\infty} a_k^3$.

(c) As a counterexample, we can take any conditionally converging series where the convergence of the terms to zero is “relatively slow”. For example, one can take $a_k = (-1)^k/\sqrt{k}$, then $\sum_{k=1}^{\infty} a_k$ converges as an alternating series with $|a_k|$ monotone and tending to zero, but $\sum_{k=1}^{\infty} a_k^2 = \sum_{k=1}^{\infty} 1/k$ diverges.

3.3. (a) Let $f : [a, b] \to \mathbb{R}$ be continuous, and let $F \subseteq [a, b]$ be closed. Show that the image $f(F) = \{y = f(x) \mid x \in F\}$ is also closed.

(b) Let $(a, b) \subset \mathbb{R}$ be a bounded interval, and let $f : (a, b) \to \mathbb{R}$ be continuous and bounded. Does this imply that f is uniformly continuous?

(c) ($\star/2$) Let $E \subset \mathbb{R}$ be bounded, and let $f, g : E \to \mathbb{R}$ be uniformly continuous. Is it true that fg is uniformly continuous?

Solution:

(a) According to Exercise 2.1(b), a set is closed if it contains all its accumulation points. Let $y \in \mathbb{R}$ be an accumulation point of $f(F)$, then for every $k \in \mathbb{N}$ there exists $y_k \in f(F)$ such that $|y - y_k| < 1/k$. Note that though some y_k may appear not once in the sequence $\{y_k\}$, every y_k shows up finitely many times. Consider the points $x_k \in F$ such that $f(x_k) = y_k$. The points $\{x_k\}$ compose a bounded sequence, so we can take a subsequence $\{x_{k_i}\}$ converging to some x, where $x \in F$ since F is closed. Then, as f is continuous,

$$f(x) = f(\lim_{i\to\infty} x_{k_i}) = \lim_{i\to\infty} f(x_{k_i}) = \lim_{i\to\infty} y_{k_i} = y,$$

which implies that $y \in f(F)$.

(b) We construct a counterexample as a "highly oscillating" function. Consider any open interval, say $E = (0, 1)$, and a function f on E defined as $f(x) = \cos(\frac{\pi}{x})$. Function f is continuous on E as a composition of continuous functions. Observe that for every $k \in \mathbb{N}$ we have $f(\frac{1}{4k+2}) = \cos(2\pi(2k + 1)) = 1$, and $f(\frac{2}{4k+2}) = \cos(\pi(2k + 1)) = -1$. Therefore, for every k we can find two points $x_1 = \frac{1}{4k+2}$ and $x_2 = \frac{2}{4k+2}$ at distance $1/(4k+2)$ such that $|f(x_1)-f(x_2)| = 2$. Thus, if we take arbitrary $\varepsilon < 2$ (say, $\varepsilon = 1$), then for every $\delta > 0$ there exist $x_1, x_2 \in E$ such that $|x_1 - x_2| < \delta$ and $|f(x_1) - f(x_2)| > \varepsilon$, which implies (by definition) that f is not uniformly continuous.

3.4. Let $f : \mathbb{R} \to \mathbb{R}$ be a monotone function. Show that the *discontinuity set* D of f, i.e. $D = \{x \in \mathbb{R} \,|\, f \text{ is not continuous at } x\}$, is either countable or empty.

Solution:

Without loss of generality, assume that f is increasing. According to an exercise from lectures, f has both left and right limits $f(x_-)$ and $f(x_+)$ at every point $x \in \mathbb{R}$. Thus, for every point $z \in D$, we have an open non-empty interval $(f(z_-), f(z_+))$, and all these intervals are disjoint as for $y < z$ we have $f(y_+) \le f(z_-)$. Taking a rational number in every such interval, we construct an injective map from D to $\mathbb{Q}$, which implies countability of D.

3.5. Which of the numbers $1/2$, $2/3$, $3/4$ belong to the Cantor set?

Solution:

$1/2 \notin C$ since $1/2 \notin C_1$.

$2/3$ is an endpoint of one of the two intervals of C_1. Observe that, by construction of sets C_n, every endpoint of every C_k belongs to C, so we deduce that $2/3 \in C$. (Alternatively, we can note that $2/3$ has ternary expression $2/3 = 0.2$, so it must belong to C.)

Finally, observe that $3/4$ divides $C_0 = [0, 1]$ in proportion $3 : 1$, and the component $[2/3, 1]$ of C_1 in proportion $1 : 3$. Now, applying induction, we deduce that it will divide the corresponding parts of C_{2k} as $3 : 1$ and the corresponding parts of C_{2k+1} as $1 : 3$, so it will belong to every C_n, and thus to C.

Alternatively, an easy computation shows that the ternary expression of $3/4$ is $0,20202020...$ Indeed,

$$\sum_{k=0}^{\infty} \frac{2}{3^{2k+1}} = \frac{2}{3}\sum_{k=0}^{\infty} \frac{1}{3^{2k}} = \frac{2}{3}\sum_{k=0}^{\infty} \frac{1}{9^k} = \frac{2}{3}\frac{1}{1-\frac{1}{9}} = \frac{2}{3}\frac{9}{8} = \frac{3}{4}.$$

3.6. (a) Let $m^*(A) = 0$. Show that $m^*(B \cup A) = m^*(B)$ for every set $B \subset \mathbb{R}$.

(b) Let $A, B \subset \mathbb{R}$ be bounded, and assume that $\sup A \le \inf B$. Show that $m^*(B\cup A) = m^*(A)+m^*(B)$.

(c) Assume that $E \subset \mathbb{R}$ has positive outer measure. Show that there exists a bounded subset of E with positive outer measure.

Solution:

(a) First, $m^*(B\cup A) \ge m^*(B)$ by monotonicity of the outer measure, so we are left to show the inverse inequality. Now, by subadditivity of the outer measure, we have $m^*(B \cup A) \le m^*(B) + m^*(A) = m^*(B)$ as $m^*(A) = 0$, so $m^*(B \cup A) = m^*(B)$.

(b) Define $E = (-\infty, \sup A)$, E is obviously measurable. Then $m^*(B\cup A) = m^*((B\cup A)\cap E)+m^*((B\cup A)\cap E^c)$. Observe that $(B \cup A) \cap E^c = B \cup \{\sup A\}$, and $(B \cup A) \cap E = A \setminus \{\sup A\}$, so

$$m^*(B\cup A) = m^*((B\cup A)\cap E)+m^*((B\cup A)\cap E^c) = m^*(B\cup\{\sup A\})+m^*(A\setminus\{\sup A\}) = m^*(B)+m^*(A)$$

by (a).

(c) Consider the sets $E_k = E \cap [-k, k]$ for every $k \in \mathbb{N}$. Then E is a union of all E_k, $k \in \mathbb{N}$, and all E_k are bounded. If we assume that the outer measure of every E_k is zero, then the outer measure of E is also zero as of a countable union of sets of outer measure zero, so we obtain a contradiction.