Durham University
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Analysis III/IV, Homework 3 (Weeks 5–6)

Due date: Thursday, November 23.

Starred problems are **more difficult** and are **not for submission**.

Series and continuous functions. Outer measure.

3.1. Prove Proposition 2.14: a series $\sum_{k=1}^{\infty} a_k$ converges if and only if

$\forall \varepsilon > 0\ \exists N \in \mathbb{N}$ such that $\forall n \geq N$ and $\forall m \in \mathbb{N}$ one has $\left|\sum_{k=n}^{n+m} a_k\right| < \varepsilon$.

3.2. (a) Show that if $\sum_{k=1}^{\infty} a_k$ converges then $\lim_{k\to\infty} a_k = 0$.

(b) Let $a_k \geq 0$ for all $k \in \mathbb{N}$. Show that if $\sum_{k=1}^{\infty} a_k$ converges then $\sum_{k=1}^{\infty} a_k^3$ also converges.

(c) Does the convergence of $\sum_{k=1}^{\infty} a_k$ imply the convergence of $\sum_{k=1}^{\infty} a_k^2$?

(d) ($\star$/2) Is the assertion of (b) true without the assumption of non-negativity of all a_k?

3.3. (a) Let $f : [a, b] \to \mathbb{R}$ be continuous, and let $F \subseteq [a, b]$ be closed. Show that the image $f(F) = \{y = f(x) \,|\, x \in F\}$ is also closed.

(b) Let $(a, b) \subset \mathbb{R}$ be a bounded interval, and let $f : (a, b) \to \mathbb{R}$ be continuous and bounded. Does this imply that f is uniformly continuous?

(c) ($\star$/2) Let $E \subset \mathbb{R}$ be bounded, and let $f, g : E \to \mathbb{R}$ be uniformly continuous. Is it true that fg is uniformly continuous?

3.4. Let $f : \mathbb{R} \to \mathbb{R}$ be a monotone function. Show that the *discontinuity set* D of f, i.e. $D = \{x \in \mathbb{R} \,|\, f \text{ is not continuous at } x\}$, is either countable or empty.

3.5. Which of the numbers 1/2, 2/3, 3/4 belong to the Cantor set?

3.6. (a) Let $m^*(A) = 0$. Show that $m^*(B \cup A) = m^*(B)$ for every set $B \subset \mathbb{R}$.

(b) Let $A, B \subset \mathbb{R}$ be bounded, and assume that $\sup A \leq \inf B$. Show that $m^*(B \cup A) = m^*(A) + m^*(B)$.

(c) Assume that $E \subset \mathbb{R}$ has positive outer measure. Show that there exists a bounded subset of E with positive outer measure.