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Analysis III/IV, Solutions 4 (Weeks 7–8)

Starred problems are **more difficult** and are **not for submission**.

Outer measure. Measurable sets

All sets below are subsets of $\mathbb{R}$.

4.1. Let E be bounded. Show that there exists a countable intersection G of open sets such that $E \subseteq G$ and $m^*(G) = m^*(E)$.

Solution:

First, observe that $m^*(E) < \infty$ since E is bounded. By the definition of the outer measure, for every $n \in \mathbb{N}$ there exists a countable collection of open intervals $\{I_k^n\}$ containing E such that $\sum\limits_{k=1}^{\infty} l(I_k^n) < m^*(E) + \frac{1}{n}$. Define $U_n = \bigcup\limits_{k=1}^{\infty} I_k^n$.Then $G = \bigcap\limits_{k=1}^{\infty} U_n$ is the required set. Indeed, G is a countable intersection of open sets U_n, it contains E (as every U_n contains E), so $m^*(G) \geq m^*(E)$ by monotonicity of the outer measure. On the other hand, for every $n \in \mathbb{N}$ $m^*(G) \leq m^*(U_n) < m^*(E) + 1/n$, so $m^*(G) \leq m^*(E)$.

4.2. Show that if E_1 and E_2 are measurable, then

$$m(E_1 \cup E_2) + m(E_1 \cap E_2) = m(E_1) + m(E_2).$$

Solution:

If any of the E_1 and E_2 has infinite measure, then the both sides are infinite. Thus, we can assume that both sets have finite measure (and thus all the sets involved as well). Observe that $E_1 \cup E_2$ is a disjoint union of $E_1 \setminus E_2$, $E_2 \setminus E_1$ and $E_1 \cap E_2$. Thus, by additivity of Lebesgue measure, we have

$$m(E_1 \cup E_2) + m(E_1 \cap E_2) = m(E_1 \setminus E_2) + m(E_1 \cap E_2) + m(E_2 \setminus E_1) + m(E_1 \cap E_2) = m(E_1) + m(E_2),$$

since E_1 is a disjoint union of $E_1 \setminus E_2$ and $E_1 \cap E_2$.

4.3. Let E have finite outer measure. Show that if E is not measurable, then there exists an open set U such that $E \subseteq U$ and

$$m^*(U \setminus E) > m^*(U) - m^*(E).$$

Solution:

By Theorem 3.19, a set A is measurable if and only if for every positive ε there exists an open set U such that $m^*(U \setminus A) < \varepsilon$. Therefore, non-measurability of E implies that there exists $\varepsilon_0 > 0$ such that for every open U containing E one has $m^*(U \setminus E) > \varepsilon_0$. As in the Exercise 4.1, take an open set U such that $E \subset U$ and $m^*(U) < m^*(E) + \varepsilon_0$. Then

$$m^*(U \setminus E) > \varepsilon_0 > m^*(U) - m^*(E).$$

4.4. A set $A \subseteq \mathbb{R}$ is called *nowhere dense* if every non-empty open $U \subseteq \mathbb{R}$ has an open non-empty subset $U_0 \subseteq U$ such that $U_0 \cap A = \emptyset$.

(a) Show that a subset of a nowhere dense set is also nowhere dense.

(b) Show that a finite union of nowhere dense sets is nowhere dense.

(c) Is a countable union of nowhere dense sets always nowhere dense?

(d) Which of the following sets are nowhere dense: $\mathbb{Z}$; $[0,1]$; $\{1/n \mid n \in \mathbb{N}\} \cup \{0\}$; $\mathbb{Q}$?

(e) Show that the Cantor set is nowhere dense.

(f) ($\star$) Is it true that every nowhere dense set has measure zero?

(g) ($\star$) Is it possible to split a closed interval into a countable union of disjoint nowhere dense sets?

Solution:

(a) This follows from the definition: given open U, the same $U_0 \subseteq U$ which does not intersect A does not intersect any its subset either.

(b) It is sufficient to prove that a union of two nowhere dense sets A_1 and A_2 is nowhere dense: we then can use induction to extend the result to any finite number.

Let U be open. We want to find an open subset of U not intersecting $A_1 \cup A_2$. Since A_1 is nowhere dense, there exists an open $U_1 \subseteq U$ not intersecting A_1. Since A_2 is nowhere dense and U_1 is open, there exists an open $U_2 \subseteq U_1$ not intersecting A_2. Thus, $U_2 \cap A_2 = \emptyset$, and $U_2 \cap A_1 = \emptyset$ since $U_1 \cap A_1 = \emptyset$ and $U_2 \subseteq U_1$. Therefore, we found open $U_2 \subseteq U$ such that $U_2 \cap (A_1 \cup A_2) = \emptyset$, so $A_1 \cup A_2$ is nowhere dense.

(c) The easiest counterexample is $\mathbb{Q}$. Of course it is not nowhere dense (since every open interval, and thus every open set, contains a rational number), and it is a countable union of a single point sets (which are obviously nowhere dense: for every open U the set $U \setminus \{x\}$ is open and infinite, and thus non-empty).

(d) $\mathbb{Z}$ is nowhere dense: for every open U the set $U_0 = U \setminus \mathbb{Z}$ is open and infinite, and thus nonempty. Alternatively, we could assume that there exists $n \in U \cap \mathbb{Z}$ (if it does not then take $U_0 = U$), then U contains an interval $(n - \varepsilon, n + \varepsilon)$, so we can take $U_0 = (n, \min(n + \varepsilon, n + 1))$.

$[0, 1]$ is not nowhere dense: it contains an open interval $(0, 1)$, and thus for $U = (0, 1)$ the set $U \setminus [0, 1]$ is empty.

The set $A = \{1/n \mid n \in \mathbb{N}\} \cup \{0\}$ is nowhere dense: it is closed and countable, so for every open U the set $U_0 = U \setminus A$ is open and uncountable, and thus nonempty. Alternatively, one could use arguments similar to ones for $\mathbb{Z}$.

Finally, $\mathbb{Q}$ is not nowhere dense, see the exercise above.

(e) Denote by G the complement of C in $[0, 1]$. We know that G is open and has measure 1.

Take any open U. If $U \setminus (0, 1)$ is not empty, then we can take an open subset $U_0 \subset U \setminus (0, 1)$ and we are done, so we can assume that $U \subseteq (0, 1)$. Denote $U_0 = U \cap G$. U_0 is open and has empty intersection with C (as it is a subset of G), so we are only left to prove that U_0 is not empty.

Suppose U_0 is empty, this means that U and G are disjoint. Then $m(U \cup G) = m(U) + m(G) = m(u) + 1 > 1$ since every non-empty open set has positive measure (and both sets are measurable). On the other hand, $U \cup G \subseteq (0, 1)$, so $m(U \cup G) \leq 1$, which leads to a contradiction.

Actually, the reasoning above proves a stronger statement: every *closed* set of *measure zero* is nowhere dense. Note that both assumptions are essential.