Durham University
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Michaelmas 2017

Analysis III/IV, Homework 4 (Weeks 7–8)

Due date: Thursday, December 7.

Starred problems are **more difficult** and are **not for submission**.

Outer measure. Measurable sets

All sets below are subsets of $\mathbb{R}$.

4.1. Let E be bounded. Show that there exists a countable intersection G of open sets such that $E \subseteq G$ and $m^*(G) = m^*(E)$.

4.2. Show that if E_1 and E_2 are measurable, then

$$m(E_1 \cup E_2) + m(E_1 \cap E_2) = m(E_1) + m(E_2).$$

4.3. Let E have finite outer measure. Show that if E is not measurable, then there exists an open set U such that $E \subseteq U$ and

$$m^*(U \setminus E) > m^*(U) - m^*(E).$$

4.4. A set $A \subseteq \mathbb{R}$ is called *nowhere dense* if every open $U \subseteq \mathbb{R}$ has an open non-empty subset $U_0 \subseteq U$ such that $U_0 \cap A = \emptyset$.

(a) Show that a subset of a nowhere dense set is also nowhere dense.

(b) Show that a finite union of nowhere dense sets is nowhere dense.

(c) Is a countable union of nowhere dense sets always nowhere dense?

(d) Which of the following sets are nowhere dense: $\mathbb{Z}$; $[0, 1]$; $\{1/n \,|\, n \in \mathbb{N}\} \cup \{0\}$; $\mathbb{Q}$?

(e) Show that the Cantor set is nowhere dense.

(f) ($\star$) Is it true that every nowhere dense set has measure zero?

(g) ($\star$) Is it possible to split a closed interval into a countable union of disjoint nowhere dense sets?