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Michaelmas 2017

Analysis III/IV, Solutions 5 (Weeks 9–10)

Measurable functions

All sets below are subsets of $\mathbb{R}$.

5.1. Let E be measurable, and let $f : E \to \mathbb{R}$ be monotone. Show that f is measurable.

Solution:

We may assume that f is increasing. Take any $c \in \mathbb{R}$, and let $b = \sup\{x \in E \mid f(x) \le c\} \in \mathbb{R} \cup \{\infty\}$. Then the full preimage of $(-\infty, c]$ is the intersection $E \cap (-\infty, b)$ (or $E \cap (-\infty, b]$) which is measurable as an intersection of two measurable sets. Thus, f is measurable.

5.2. Give an example of a non-measurable function $f : \mathbb{R} \to \mathbb{R}$ such that all sets $\{x \in \mathbb{R} \mid f(x) = c\}$ are measurable.
Hint: construct a non-measurable function $f : \mathbb{R} \to \mathbb{R}$ such that every set $\{x \in \mathbb{R} \mid f(x) = c\}$ consists of at most one point.

Solution:

Let $X \subset [0, 1]$ be non-measurable. Define

$$f(x) = \begin{cases} x & \text{if } x < 0; \\ x & \text{if } x \in [0,1] \setminus X; \\ x+1 & \text{if } x \in X; \\ x+1 & \text{if } x > 1. \end{cases}$$

This function takes every value at most once, and the preimage of an interval $(1, \infty)$ is $X \cup (1, \infty)$ which is non-measurable.

5.3. Assume that a function $f : \mathbb{R} \to \mathbb{R}$ is measurable and that $f(x) > 0$ for all $x \in \mathbb{R}$. Prove that $g : \mathbb{R} \to \mathbb{R}, g(x) = \frac{1}{f(x)}$ is measurable.

Solution:

For $c \le 0$, the preimage of $(-\infty, c)$ is empty. Given $c > 0$, $g(x) < c$ is equivalent to $f(x) > \frac{1}{c}$, and the set of $x \in \mathbb{R}$ such that $f(x) > \frac{1}{c}$ is measurable due to measurability of f.

5.4. Let $f, g : [a, b] \to \mathbb{R}$ be continuous.

(a) Show that if $f = g$ a.e. on $[a, b]$ then $f \equiv g$.

(b) Is the assertion of (a) true if f, g are defined on an arbitrary measurable subset of $\mathbb{R}$?

Solution:

(a) Suppose there exists $x \in [a, b]$ such that $f(x) \ne g(x)$, we may assume $f(x) < g(x)$. By continuity of f and g, there exists $\delta > 0$ such that for every x' satisfying $|x - x'| < \delta$ one has $|f(x) - f(x')| < (g(x) - f(x))/2$ and $|g(x) - g(x')| < (g(x) - f(x))/2$. Then, by the triangle inequality, $f(x') < g(x')$ for every x' satisfying $|x - x'| < \delta$, so $f \ne g$ on an interval, which of course has a positive measure.

(b) Take any E of measure zero. Then any two continuous functions are equal a.e. on E, so the assertion of (a) is obviously wrong. Though, it is a good exercise to formulate necessary and sufficient conditions for E to satisfy the assertion of (a).

5.5. Let $f = \sum_{k=1}^{n} c_k \chi_{E_k}$ be a simple function, where $c_k \in \mathbb{R}$ and χ_{E_k} are indicator functions of some mutually disjoint sets $E_k \subseteq \mathbb{R}$. Show that f is measurable if and only if every E_k is measurable.

Solution:

Note that, in constrast to the definition given in lectures, here "a simple function" means just a function taking finitely many values.

If the function is measurable, then every E_k is measurable as a preimage of a single point interval. Conversely, the preimage of $(-\infty, c)$ is precisely the union of those E_k for which $c_k < c$, and thus it is measurable in the case all E_k are measurable.

5.6. Let $f : E \to \mathbb{R}$, E is measurable. Define $g : \mathbb{R} \to \mathbb{R}$ by $g(x) = 0$ for $x \notin E$ and $g(x) = f(x)$ if $x \in E$. Show that f is measurable if and only if g is measurable.

Solution:

Given $c \in \mathbb{R}$, the preimage $g^{-1}((-\infty, c))$ can be expressed in the following way:

$$g^{-1}((-\infty, c)) = \begin{cases} f^{-1}((-\infty, c)) & \text{if } c \leq 0; \\ f^{-1}((-\infty, c)) \cup (\mathbb{R} \setminus E) & \text{if } c > 0. \end{cases}$$

Since E (and thus $\mathbb{R} \setminus E$) is measurable, the measurability of the sets above is equivalent to the measurability of f.

5.7. (a) Let $f, g : E \to \mathbb{R}$ be measurable functions. Show that $\max(f, g)$ is measurable.

(b) Let $\{f_n : E \to \mathbb{R}\}$ be a sequence of measurable functions, define a function $\sup f_n : E \to \mathbb{R}$ by $(\sup f_n)(x) = \sup\{f_n(x) \mid n \in \mathbb{N}\}$. Is $\sup f_n$ always measurable?

Solution:

(a) Given $c \in \mathbb{R}$, $\max(f, g)(x) < c$ is equivalent to $f(x) < c$ and $g(x) < c$ simultaneously. Thus, $\{x \in \mathbb{R} \mid \max(f, g)(x) < c\} = \{x \in \mathbb{R} \mid f(x) < c\} \cap \{x \in \mathbb{R} \mid g(x) < c\}$ and thus is measurable as an intersection of two measurable sets.

(b) This is true, and the proof is almost identical to (a):

$$\{x \in \mathbb{R} \mid (\sup f_n)(x) \leq c\} = \bigcap_{n=1}^{\infty} \{x \in \mathbb{R} \mid f_n(x) \leq c\}$$

and thus is measurable.