Durham University
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Analysis III/IV, Homework 5 (Weeks 9–10)

Measurable functions

All sets below are subsets of $\mathbb{R}$.

5.1. Let E be measurable, and let $f : E \to \mathbb{R}$ be monotone. Show that f is measurable.

5.2. Give an example of a non-measurable function $f : \mathbb{R} \to \mathbb{R}$ such that all sets $\{x \in \mathbb{R} \mid f(x) = c\}$ are measurable.
Hint: construct a non-measurable function $f : \mathbb{R} \to \mathbb{R}$ such that every set $\{x \in \mathbb{R} \mid f(x) = c\}$ consists of at most one point.

5.3. Assume that a function $f : \mathbb{R} \to \mathbb{R}$ is measurable and that $f(x) > 0$ for all $x \in \mathbb{R}$. Prove that $g : \mathbb{R} \to \mathbb{R}, g(x) = \frac{1}{f(x)}$ is measurable.

5.4. Let $f, g : [a, b] \to \mathbb{R}$ be continuous.

(a) Show that if $f = g$ a.e. on $[a, b]$ then $f \equiv g$.

(b) Is the assertion of (a) true if f, g are defined on an arbitrary measurable subset of $\mathbb{R}$?

5.5. Let $f = \sum_{k=1}^{n} c_k \chi_{E_k}$ be a simple function, where $c_k \in \mathbb{R}$ and χ_{E_k} are indicator functions of some mutually disjoint sets $E_k \subseteq \mathbb{R}$. Show that f is measurable if and only if every E_k is measurable.

5.6. Let $f : E \to \mathbb{R}$, E is measurable. Define $g : \mathbb{R} \to \mathbb{R}$ by $g(x) = 0$ for $x \notin E$ and $g(x) = f(x)$ if $x \in E$. Show that f is measurable if and only if g is measurable.

5.7. (a) Let $f, g : E \to \mathbb{R}$ be measurable functions. Show that $\max(f, g)$ is measurable.

(b) Let $\{f_n : E \to \mathbb{R}\}$ be a sequence of measurable functions, define a function $\sup f_n : E \to \mathbb{R}$ by $(\sup f_n)(x) = \sup\{f_n(x) \mid n \in \mathbb{N}\}$. Is $\sup f_n$ always measurable?