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Analysis III/IV

Please see Sections I.1–I.3 of *Real Analysis* by Royden and Fitzpatrick for details.

1 Real numbers

1.1 Ordered fields

Definition 1.1 (Reminder)**.** Definition of a field: axioms of a field.

Exercise. Let $\mathbb{F}$ be a field.

(a) Show that $0 \in \mathbb{F}$ is unique.

(b) Show that $1 \in \mathbb{F}$ is unique.

(c) Show that for every $a \in \mathbb{F}$ its negative $-a$ and inverse a^{-1} are unique.

(d) Show that $(-1) \cdot a = -a$ and $a \cdot 0 = 0$ for every $a \in \mathbb{F}$.

Example 1.2. Field $\mathbb{F}_2 = \{0, 1\}$; $\mathbb{Q}$.

Definition 1.3. An *ordered field* is a field $\mathbb{F}$ with a subset $\mathbb{P} \subset \mathbb{F}$ (elements of which are called *positive numbers*) satisfying the following two properties:

(1) If $a, b \in \mathbb{P}$ then $a + b \in \mathbb{P}$ and $a \cdot b \in \mathbb{P}$.

(2) For every $a \in \mathbb{F}$ exactly one of the following holds: either $a \in \mathbb{P}$, or $a = 0$, or $-a \in \mathbb{P}$.

We say that $a > b$ (or $b < a$) if $a - b \in \mathbb{P}$, and $a \geq b$ (or $b \leq a$) if $a > b$ or $a = b$.

Exercise. Show that $1 \in \mathbb{P}$.

Definition 1.4 (Reminder)**.** Open and closed intervals.

Definition 1.5 (Reminder)**.** Upper and lower bounds of a set, supremum and infimum. Bounded set.

Definition 1.6 (Completeness Axiom)**.** $\mathbb{R}$ is an ordered field satisfying the following *completeness axiom*: every bounded from above subset of $\mathbb{R}$ has a supremum.

Exercise. Show that every bounded from below subset A of $\mathbb{R}$ has infimum, and $\inf A = -\sup(-A)$.

Definition 1.7 (Reminder)**.** Metric space.

Example 1.8. $d(x, y) = |x - y|$ is a metric on $\mathbb{R}$.

1.2 $\mathbb{N}$, $\mathbb{Z}$ and $\mathbb{Q}$

Definition 1.9. $A \subseteq \mathbb{R}$ is *inductive* if

(1) $1 \in A$;

(2) $\forall a \in A\ a+1 \in A$.

Example 1.10. $\mathbb{R}$ is inductive.

Definition 1.11. The set of *natural numbers* $\mathbb{N}$ is the intersection of all inductive subsets of $\mathbb{R}$.

Properties of $\mathbb{N}$:

- $\mathbb{N}$ is not empty;
- $\mathbb{N}$ is inductive;
- $\mathbb{N} \subseteq \mathbb{P}$.

Theorem 1.12 (Mathematical Induction)**.** *Let for every $n \in \mathbb{N}$ $S(n)$ denote some statement. Suppose that $S(1)$ is true, and for every $k \in \mathbb{N}$ the statement $S(k)$ implies $S(k+1)$. Then $S(n)$ is true for every $n \in \mathbb{N}$.*

Exercise 1.13. (a) Let $a, b \in \mathbb{N}$. Show that $a+b \in \mathbb{N}$, $ab \in \mathbb{N}$.

(b) Let $a \in \mathbb{N}$, $a > 1$. Show that $a-1 \in \mathbb{N}$.

(c) Let $a, b \in \mathbb{N}$, $a > b$. Show that $a-b \in \mathbb{N}$.

(d) Let $n \in \mathbb{N}$. Show that there is no natural number m such that $n < m < n+1$.

Theorem 1.14. *Every non-empty subset of $\mathbb{N}$ has a smallest element.*

Proposition 1.15 (Archimedian Property)**.** *For every $a, b > 0$ there exists $n \in \mathbb{N}$ such that $na > b$.*

Definition 1.16. *Integer numbers* $\mathbb{Z} = \mathbb{N} \cup \{0\} \cup (-\mathbb{N})$, *rational numbers* $\mathbb{Q} = \{m/n \,|\, m \in \mathbb{Z},\, n \in \mathbb{N}\}$, *irrational numbers* $\mathbb{R} \setminus \mathbb{Q}$.

Example 1.17. $\mathbb{Q}$ is an ordered field.

Proposition 1.18. *For every $a, b \in \mathbb{R}$, $a < b$, there exists $c \in \mathbb{Q}$ such that $a < c < b$.*

1.3 Countable and uncountable sets

Definition 1.19.

- Two sets A and B are *equipotent* if there exists a bijection between them. Notation: $|A| = |B|$.
- A set A is *finite* if it is equipotent to a bounded subset $\{1, 2, \ldots, n\}$ of $\mathbb{N}$. Notation: $|A| = n$. A is *infinite* otherwise.
- A set A is *countably infinite* if it is equipotent to $\mathbb{N}$.
- A set A is *countable* if it is either finite or countably infinite.
- A set A is *uncountable* if it is not countable.

Exercise. Show that every ordered field is infinite.

Example 1.20. $2\mathbb{N}$ is countable.

Exercise. Show that $\mathbb{Z}$ and $\mathbb{N} \setminus \{2017\}$ are countable.

Proposition 1.21. *A non-empty subset of a countable set is countable.*

Proposition 1.22. *A non-empty set B is countable if and only if there exists a surjective map $A \to B$ for some countable set A.*

Proposition 1.23. *Let X and Y be countable. Then $X \times Y$ is countable.*

Corollary 1.24. *$\mathbb{Q}$ is countable.*

Exercise.

- A union of a finite number of countable sets is countable.
- A countable union of finite sets is countable.
- A countable union of countable sets is countable.

Proposition 1.25. *An intersection of a countable collection of closed nested intervals is not empty.*

Corollary 1.26. *Let $a, b \in \mathbb{R}$, $a < b$. Then $[a, b]$ is uncountable.*

Corollary 1.27. *$\mathbb{R}$ is uncountable.*

Exercise. Let $\{I_n\}$ be a system of closed nested intervals. Show that their intersection is a single point if and only if for every positive ε there exists an $n \in \mathbb{N}$ such that the length of I_n is less than ε.

Definition. Given a set X, the *set of all subsets* of X is denoted by $P(X)$ or 2^X.

Exercise. Show that $P(\mathbb{N})$ is uncountable.

1.4 Open and closed subsets of $\mathbb{R}$

All sets are subsets of $\mathbb{R}$.

Definition 1.28 (Reminder)**.** A subset $A \subseteq \mathbb{R}$ is *open* if $\forall a \in A\, \exists \varepsilon > 0$ s.t. $(a - \varepsilon, a + \varepsilon) \subseteq A$.

Example 1.29. Open interval is open.

Example 1.30.

- $\mathbb{R}$ and $\emptyset$ are open.
- A finite intersection of open sets is open.
- A union of any collection of open sets is open.

Example 1.31. A countable intersection of open sets may not be open: $\bigcap_{n \in \mathbb{N}} (-1/n, 1/n) = \{0\}$.

Proposition 1.32. *Every non-empty open set is a union of a countable set of mutually disjoint open intervals.*

Definition 1.33. Let A be a set. $x \in \mathbb{R}$ is a *closure point* (or a *point of closure*) of A if for $\forall \varepsilon > 0$ $(x - \varepsilon, x + \varepsilon) \cap A \neq \emptyset$. The *closure* $\overline{A}$ of A is the set of all closure points of A. A is *closed* if $\overline{A} = A$.

Proposition 1.34. *For every set A its closure $\overline{A}$ is closed. Moreover, $\overline{A}$ is a subset of any closed set containing A.*

Proposition 1.35. *A is closed if and only if its complement $\mathbb{R} \setminus A$ is open.*

Corollary 1.36. *A union of a finite number of closed sets is closed. An intersection of any collection of closed sets is closed.*

Exercise (1.37'). A closed bounded set contains its infimum and supremum.

Definition 1.37 (Reminder)**.** An *open cover* of a set A is a collection $\{U_\lambda\}_{\lambda\in\Lambda}$ of open sets such that $A \subseteq \bigcup_{\lambda\in\Lambda} U_\lambda$. A is *compact* if every open cover of A has a finite subcover.

Theorem 1.38 (Heine – Borel Theorem)**.** *A subset of $\mathbb{R}$ is compact if and only if it is closed and bounded.*

1.5 Borel sets in $\mathbb{R}$

Proposition 1.39 (Nested Sets Theorem)**.** *An intersection of a countable collection of closed nested sets is non-empty.*

Definition 1.40. Let X be a set. A collection $\mathcal{A}$ of subsets of X is called a *σ-algebra* of subsets of X is it satisfies the following:

(1) $\emptyset \in \mathcal{A}$;

(2) if $A \in \mathcal{A}$ then $X \setminus A \in \mathcal{A}$;

(3) a union of a countable collection of elements of $\mathcal{A}$ also belongs to $\mathcal{A}$.

Definition 1.41. *Borel sets* are elements of the smallest σ-algebra $\mathcal{B}$ containing all open sets.

Example. $\mathcal{B}$ contains: all open and closed sets; countable unions of closed sets.

Exercise. (a) Every open set is a countable union of closed sets.

(b) $\mathcal{B}$ is the smallest σ-algebra containing all closed sets.

2 Sequences and continuity in $\mathbb{R}$

2.1 Sequences in $\mathbb{R}$

Definition 2.1 (Reminder)**.** A *sequence* $\{a_i\}$ in $\mathbb{R}$ is a map $f : \mathbb{N} \to \mathbb{R}$, $f(i) = a_i$. A sequence is *bounded* if the set of its elements is bounded. A sequence is *increasing* (*decreasing*) if $a_i \leq a_{i+1}$ ($a_i \geq a_{i+1}$ respectively) for every $i \in \mathbb{N}$, *monotone* if it is either increasing or decreasing.

Definition 2.2 (Reminder)**.** A sequence $\{a_n\}$ *converges* to $a \in \mathbb{R}$ if $\forall \varepsilon > 0\, \exists N \in \mathbb{N}$ such that $|a_n - a| < \varepsilon$ $\forall\, n > N$. Notation: $\lim_{n\to\infty} a_n = a$.

Exercise. Let $\{a_n\}$ converge. Then the sequence is bounded, and limit is unique.

Proposition 2.3 (Reminder)**.** *Let $\{a_n\}$ be monotone. Then it converges if and only if it is bounded.*

Theorem 2.4 (Bolzano – Weierstrass Theorem)**.** *Every bounded sequence in $\mathbb{R}$ has a converging subsequence.*

Exercise. $\lim_{n\to\infty} a_n = a$ if and only if every subsequence of $\{a_n\}$ converges to a.

Definition 2.5. a is an *accumulation point* of $\{a_n\}$ if $\forall \varepsilon > 0\, \forall N \in \mathbb{N}\, \exists n > N$ s.t. $|a_n - a| < \varepsilon$.

Exercise. Show that a is an accumulation point of $\{a_n\}$ if and only if there exists a subsequence of $\{a_n\}$ converging to a.

Definition 2.6 (Reminder)**.** $\{a_n\}$ is a *Cauchy sequence* if $\forall\varepsilon > 0\,\exists N \in \mathbb{N}$ s.t. $|a_n - a_m| < \varepsilon\,\forall\, m, n > N$.

Theorem 2.7 (Reminder)**.** *A sequence of real numbers converges if and only if it is a Cauchy sequence.*

Definition 2.8 (Reminder)**.** Convergence to $\pm\infty$. sup and inf of unbounded sets.

Definition 2.9. A *limit superior* of $\{a_n\}$ is defined by $\limsup a_n = \lim_{n\to\infty} \sup\{a_k \mid k \geq n\}$.
A *limit inferior* of $\{a_n\}$ is defined by $\liminf a_n = \lim_{n\to\infty} \inf\{a_k \mid k \geq n\}$.

Example 2.10.
- If $\{a_n\}$ is unbounded from above (below) then $\limsup a_n = +\infty$ ($\liminf a_n = -\infty$ respectively).
- Let $a_n = (-1)^n$. Then $\limsup a_n = 1$, $\liminf a_n = -1$.
- Let $\{a_n\}$ converge. Then $\limsup a_n = \liminf a_n = \lim a_n$

Proposition 2.11. *Both limit superior and limit inferior always exist, and they are equal to the largest (smallest, respectively) accumulation point of $\{a_n\}$.*

Example 2.12. Let $P(z) = \sum_{n=0}^{\infty} a_n z^n$ be a complex power series, $a_n \neq 0$. Then its convergence radius R exists and $\frac{1}{R} = \limsup \left|\frac{a_{n+1}}{a_n}\right|$.

Exercise.
- Show that $\limsup a_n = -\liminf(-a_n)$.
- Let $a_n \leq b_n$ for every $n \in \mathbb{N}$. Show that $\limsup a_n \leq \limsup b_n$.

Definition 2.13 (Reminder)**.** Converging series.

Proposition 2.14. *Show that $\sum a_k$ converges if and only if $\forall\varepsilon > 0\,\exists N \in \mathbb{N}$ such that $\left|\sum_{k=n}^{n+m} a_k\right| < \varepsilon$ $\forall\, n > N, m \geq 0$.*

2.2 Continuous functions

Definition 2.15 (Reminder)**.** Let $E \subseteq \mathbb{R}$. A function $f : E \to \mathbb{R}$ is *continuous at $x \in E$* if $\forall\varepsilon > 0\,\exists\delta > 0$ s.t. $|f(x') - f(x)| < \varepsilon\,\forall\, x' \in E$ satisfying $|x' - x| < \delta$. f is *continuous on E* if it continuous at every $x \in E$.

Proposition 2.16 (Reminder)**.** *f is continuous at x if and only if for every sequence $\{x_n\}$ of elements of E converging to x the sequence $\{f(x_n)\}$ converges to $f(x)$.*

Proposition 2.17. *f is continuous on E if and only if for every open $A \subseteq \mathbb{R}$ there exists an open $U \subseteq \mathbb{R}$ such that $f^{-1}(A) = E \cap U$ (where $f^{-1}(A)$ is the full preimage of A).*

Example 2.18. $f : \mathbb{R} \setminus \mathbb{Q} \to \mathbb{R}$, $f(x) = 1/x$, is continuous on its domain.

Theorem 2.19 (Heine – Borel Theorem)**.** *A continuous $\mathbb{R}$-valued function on a compact set takes its minimal and maximal values.*

Theorem 2.20 (Intermediate Value Theorem)**.** *Let $f : [a, b] \to \mathbb{R}$ be continuous, $f(a) < c < f(b)$. Then there exists $x_0 \in (a, b)$ such that $f(x_0) = c$.*

Definition 2.21 (Reminder)**.** $f : E \to \mathbb{R}$ is *uniformly continuous* on E if $\forall\varepsilon > 0\,\exists\delta > 0$ s.t. $|f(x') - f(x)| < \varepsilon$ $\forall\, x, x' \in E$ satisfying $|x' - x| < \delta$.

Theorem 2.22. *A continuous function on a compact set is uniformly continuous.*

Definition 2.23 (Reminder)**.** Monotone function.

Definition 2.24 (Reminder)**.** One-sided limits.

Exercise. f is continuous at x if and only if both one-sided limits exist and are equal to $f(x)$.

Exercise. A monotone function has both one-sided limits at every point.

Theorem 2.25. *A monotone function defined on an interval is continuous if and only if its image (range) is an interval.*

3 Lebesgue measure

3.1 Outer measure

Definition 3.1. Let $A \subseteq \mathbb{R}$. For every countable cover $I = \{I_k\}_{k\in\mathbb{N}}$ of A by open intervals I_k consider the sum $\sum_{k=1}^{\infty} l(I_k)$, where $l(I_k)$ is the length of I_k (both length and sum may be infinite). The *outer measure* $m^*(A)$ is defined by

$$m^*(A) = \inf_{I \supseteq A} \{\sum_{k=1}^{\infty} l(I_k) \mid I = \bigcup_{k=1}^{\infty} I_k\}.$$

Remark 3.2. m^* is *monotone*: if $A \subseteq B$ then $m^*(A) \leq m^*(B)$.

Example 3.3. $m^*(\emptyset) = 0$; outer of a countable set is equal to zero.

Proposition 3.4. *Outer measure of an interval is equal to its length.*

Proposition 3.5. *Outer measure is* translation invariant*: $m^*(A+y) = m^*(A)$, where $A+y = \{a+y \mid a \in A\}$.*

Proposition 3.6. *Outer measure is* countably subadditive*: $m^*(\bigcup_{k=1}^{\infty} E_k) \leq \sum_{k=1}^{\infty} m^*(E_k)$.*

Example 3.7. Cantor set C, its properties: C is closed, has outer measure zero, is uncountable.

Exercise. C is *perfect*, i.e. $x \in \overline{C \setminus \{x\}}$ $\forall x \in C$ (every point of C is an accumulation point).

3.2 Lebesgue measurable sets

Notation: $\mathbb{R} \setminus E = E^c$ – complement of E.

Definition 3.8. $E \subset \mathbb{R}$ is (Lebesgue) measurable if for every set $A \subseteq \mathbb{R}$ the following holds:

$$m^*(A) = m^*(A \cap E) + m^*(A \cap E^c).$$

Proposition 3.9. *E is measurable if and only if for every set $A \subseteq \mathbb{R}$ $m^*(A) \geq m^*(A \cap E) + m^*(A \cap E^c)$.*

Proposition 3.10 (Finite additivity)**.** *Let E be measurable. Then for any $C \subseteq \mathbb{R}$ disjoint from E $m^*(E \cup C) = m^*(E) + m^*(C)$.*

Example 3.11. Every set of outer measure zero is measurable.

Proposition 3.12. *Finite union of measurable sets is measurable.*

Corollary 3.13. *Let $\{E_k\}_{k=1}^n$ be measurable and (mutually) disjoint. Then for every set $A \subseteq \mathbb{R}$*

$$m^*(A \cap (\cup E_k)) = \sum m^*(A \cap E_k).$$

Remark 3.14. Measurability is closed under finite union, taking complement, finite intersection, taking difference.

Proposition 3.15. *A union of countably many measurable sets is measurable. In particular, measurable sets form a σ-algebra.*

Proposition 3.16. *Every interval is measurable.*

Corollary 3.17. *σ-algebra of measurable sets contains all Borel sets.*

Exercise. Every translate of a measurable set is measurable.

3.3 Approximation of measurable sets

Proposition 3.18 (Excision Property)**.** *Let $A \subseteq B \subseteq \mathbb{R}$, A is measurable, $m^*(A) < \infty$. Then $m^*(B \backslash A) = m^*(B) - m^*(A)$.*

Theorem 3.19 (Outer and inner approximation)**.** *TFAE (The Following Are Equivalent):*

(0) *E is measurable.*

(1) $\forall \varepsilon > 0 \, \exists$ *open $U \supseteq E$ such that $m^*(U \setminus E) < \varepsilon$.*

(2) *There exists a countable intersection G of open sets such that $G \supseteq E$ and $m^*(G \setminus E) = 0$.*

(3) $\forall \varepsilon > 0 \, \exists$ *closed $F \subseteq E$ such that $m^*(E \setminus F) < \varepsilon$.*

(4) *There exists a countable union $\tilde{F}$ of closed sets such that $\tilde{F} \subseteq E$ and $m^*(E \setminus \tilde{F}) = 0$.*

Proposition 3.20. *Let E be measurable of finite outer measure. Then for every $\varepsilon > 0$ there exists a finite collection of open intervals $\{I_k\}_{k=1}^n$ such that $m^*(E \triangle (\cup_{k=1}^n I_k)) < \varepsilon$ (where $A \triangle B$ is the symmetric difference of A and B).*

3.4 Countable additivity

Definition 3.21. The restriction of m^* on measurable sets is called *Lebesgue measure*, $m(E) = m^*(E)$.

Proposition 3.22. *Lebesgue measure is* countably additive*: for a countable collection $\{E_k\}_{k=1}^\infty$ of mutually disjoint sets $m(\cup E_k) = \sum m(E_k)$.*

Corollary 3.23 (Summary of properties of m)**.**

- *$m(I) = l(I)$ for any interval I;*
- *m is translation-invariant;*
- *m is countably additive.*

Notation: descending (nested) and ascending sets.

Theorem 3.24 (Continuity of measure)**.**

(1) *If $\{A_k\}_{k=1}^\infty$ is an ascending sequence of measurable sets, then $m(\cup A_k) = \lim_{k\to\infty} m(A_k)$.*

(2) *If* $\{B_k\}_{k=1}^{\infty}$ *is a decending sequence of measurable sets, then* $m(\cap B_k) = \lim_{k\to\infty} m(B_k)$*.*

Notation: *a.e.* or *almost everywhere*, i.e. on a complement to a measure zero set.

Example 3.25. Every monotone function is continuous a.e. on $\mathbb{R}$.

Theorem 3.26 (Borel – Cantelli Lemma)**.** *Let* $\{E_k\}_{k=1}^{\infty}$ *be a countable collection of measurable sets s.t.* $\sum_{k=1}^{\infty} m(E_k) < \infty$*. Then* almost all $x \in \mathbb{R}$ *belong to a finitely many of sets* E_k *(or to none of them), i.e. the set of* $x \in \mathbb{R}$ *belonging to infinitely many of sets* E_k *has measure zero.*

3.5 Non-measurable sets

Lemma 3.27. *Let* $E \subseteq \mathbb{R}$ *be measurable and bounded. Suppose there exists a countably infinite bounded set* $\Lambda \subset \mathbb{R}$ *such that all sets in the collection* $\{E + \lambda\}_{\lambda\in\Lambda}$ *are mutually disjoint. Then* $m(E) = 0$*.*

Example 3.28. Example of such E and Λ for $m(E) = 0$: $E = [0, 1] \cap \mathbb{Q}$, $\Lambda = \{1/\sqrt{p}\}_{p \text{ prime}}$.

Axiom of Choice: given a collection $\mathcal{F}$ of sets, there exists a set Λ containing exactly one element from every set of $\mathcal{F}$.

Example. A set of all sets is not a set (explain this!)

Example 3.29. Construction of a set C_E for a given set E as the set of equivalence classes modulo $\mathbb{Q}$. $C_{[0,1]}$ is non-measurable.

Theorem 3.30. *Let* E *be bounded,* $m^*(E) > 0$*. Then* C_E *is non-measurable.*

Corollary 3.31 (Vitali's Theorem)**.** *Every set of positive outer measure contains a non-measurable subset.*

Example 3.32. Cantor – Lebesgue function $\varphi : [0, 1] \to [0, 1]$, its properties: φ is continuous, increasing, locally constant a.e. (more precisely, on the complement to the Cantor set C).

Example 3.33. $\psi : [0, 1] \to [0, 2]$, $\psi(x) = \varphi(x) + x$. Properties: ψ is strictly increasing, continuous, takes $[0, 1]$ onto $[0, 2]$; $\psi(C)$ has measure 1; there exists a measurable set $E \subset C$ (i.e., $m^*(E) = 0$) such that $\psi(E)$ is non-measurable.

Exercise. (a) Let $f : [a, b] \to \mathbb{R}$ be continuous and strictly increasing. Then the inverse $f^{-1} : f([a, b]) \to [a, b]$ of f is also continuous.

(b) Let $f : [a, b] \to \mathbb{R}$ be continuous, $B \subseteq \mathbb{R}$ is a Borel set. Then the preimage $f^{-1}(B)$ is also a Borel set.

(c) Let $f : [a, b] \to \mathbb{R}$ be continuous and strictly increasing, $B \subseteq [a, b]$ is a Borel set. Then the image $f(B)$ is also a Borel set.

Corollary 3.34. *There exists a set of measure zero which is not Borel.*

4 Lebesgue measurable functions

4.1 Definition and operations

Proposition 4.1. *TFAE (The Following Are Equivalent):*

(1) $\forall c \in \mathbb{R}\, \{x \in E \,|\, f(x) > c\}$ *is measurable.*

(2) $\forall c \in \mathbb{R}\, \{x \in E \,|\, f(x) \geq c\}$ *is measurable.*

(3) $\forall c \in \mathbb{R}\, \{x \in E \,|\, f(x) < c\}$ *is measurable.*

(4) $\forall c \in \mathbb{R}\, \{x \in E \,|\, f(x) \leq c\}$ *is measurable.*

Also, any of (1)–(4) *implies*

(5) $\forall c \in \mathbb{R}\, \{x \in E \,|\, f(x) = c\}$ *is measurable.*

Definition 4.2. $f : \mathbb{E} \to \mathbb{R} \cup \{\pm\infty\}$ is *measurable* if E is measurable and f satisfies assumptions (1)–(4) of Proposition 4.1.

Example 4.3. *Indicator function* of a set $[a, b] \setminus E_0$, $E_0 \subset [a, b]$.

Example 4.4. Indicator function is measurable if and only if E_0 is measurable.

Proposition 4.5. *Let E be measurable, $f : \mathbb{E} \to \mathbb{R} \cup \{\pm\infty\}$. Then f is measurable if and only if for every open $U \subseteq \mathbb{R}$ the preimage $f^{-1}(U)$ is measurable.*

Corollary 4.6. *A continuous function on a measurable set is measurable.*

Example 4.7. A monotone function on an interval is measurable.

Proposition 4.8. *Let $f, g : \mathbb{E} \to \mathbb{R} \cup \{\pm\infty\}$, E is measurable.*

(1) *If f is measurable and $f = g$ a.e. on E, then g is measurable.*

(2) *Let $D \subset E$ be measurable. Then f is measurable if and only if its restrictions on D and $E \setminus D$ are measurable.*

Proposition 4.9. *Let $f, g : \mathbb{E} \to \mathbb{R} \cup \{\pm\infty\}$ be measurable, f, g are finite a.e. on E. Then*

(1) *for every $\alpha, \beta \in \mathbb{R}$ the function $\alpha f + \beta g$ is measurable;*

(2) *fg is measurable.*

Example 4.10. Composition of a measurable and continuous functions which is not measurable: take $E \subset C$ such that $\psi(E)$ is non-measurable (see Example 3.33), then $\chi_E \circ \psi^{-1}$ is non-measurable.

Proposition 4.11. *Let $g : E \to \mathbb{R}$ be measurable, $f : \mathbb{R} \to \mathbb{R}$ continuous. Then $f \circ g$ is measurable.*

Example 4.12. If f is measurable, then $|f|$ is measurable, $|f|^\alpha$ is measurable for $\alpha \in \mathbb{R}$.

Example 4.13. A maximum of a finite collection of measurable functions is measurable.

4.2 Pointwise limits and simple approximation

Definition 4.14 (Reminder)**.** Let $\{f_n\} : E \to \mathbb{R}$ be a sequence of functions, $A \subseteq E$, $f : E \to \mathbb{R} \cup \{\pm\infty\}$.

(1) $\{f_n\}$ converges to f *pointwise* on A if for every $x \in A$ $\lim_{n\to\infty} f_n(x) = f(x)$.

(2) $\{f_n\}$ converges to f *pointwise a.e.* on A if it converges to f pointwise on $A \setminus B$, $m(B) = 0$.

(3) $\{f_n\}$ converges to f *uniformly* on A if $\forall \varepsilon > 0 \, \exists N \in \mathbb{N}$ s.t. $|f_n(x) - f(x)| < \varepsilon \, \forall n > N \, \forall x \in A$.

Example 4.15. Let $f_n = x^n : [0, 1] \to [0, 1]$, $f \equiv 0$. Then $\{f_n\}$ does not converge to f on $[0, 1]$, converges to f on $[0, 1]$ pointwise a.e., and converges to f uniformly on $[0, r]$ for any $r \in (0, 1)$.

Proposition 4.16. *Let $\{f_n\}$ converge to f pointwise a.e. on E, and all f_n are measurable. Then f is measurable.*

Definition 4.17. A measurable function is *simple* if it takes only finitely many values.

Example 4.18.

- χ_A is simple if A is measurable.
- Let $E = \bigcup_{i=1}^n E_i$, all E_i are measurable and disjoint, and $c_i \in \mathbb{R}$, $I = 1, \ldots, n$. Then the function $f : E \to \mathbb{R}$ defined by $f(x) = \sum_{i=1}^n c_i \chi_{E_i}$ is simple.
- Conversely, every simple fiunction $f : E \to \mathbb{R}$ can be written as $f(x) = \sum_{i=1}^n c_i \chi_{E_i}$, where $E_i = f^{-1}(c_i)$.

Proposition 4.19 (Simple Approximation Lemma)**.** *Let $f : E \to \mathbb{R}$ be measurable and bounded. Then for any $\varepsilon > 0$ there exist simple functions φ_ε and ψ_ε s.t. $\varphi_\varepsilon \le f \le \psi_\varepsilon$ and $\psi_\varepsilon - \varphi_\varepsilon < \varepsilon$.*

Proposition 4.20 (Simple Approximation Theorem)**.** *Let $f : E \to \mathbb{R} \cup \{\pm\infty\}$, E is measurable. Then f is measurable if and only if there exists a sequence φ_n of simple functions on E converging to f pointwise a.e. such that $|\varphi_n| \le |f|$ for every $n \in \mathbb{N}$.*

Proposition 4.21. *Let $f : \mathbb{R} \to \mathbb{R}$ be bounded. Then f is measurable if and only if there exists a sequence of simple functions converging to f uniformly.*