Durham University
Pavel Tumarkin Michaelmas 2019

Riemannian Geometry IV, Solutions 1 (Week 1)

1.1. ($\star$) Let M be a smooth manifold of dimension m and N be a smooth manifold of dimension n. Show that the cartesian product

$$M \times N := \{(x, y) \mid x \in M, y \in N\}$$

is a smooth manifold of dimension $m + n$.

Solution:

Let $\{(U_\alpha, \varphi_\alpha)\}_{\alpha\in A}$ and $\{(\widetilde{U}_\beta, \widetilde{\varphi}_\beta)\}_{\beta\in\widetilde{A}}$ be atlases on M and N, respectively. Then an atlas of $M\times N$ is given by $\{(U_\alpha \times \widetilde{U}_\beta, \psi_{\alpha,\beta})\}_{(\alpha,\beta)\in A\times\widetilde{A}}$, where

$$\psi_{\alpha,\beta} : U_\alpha \times \widetilde{U}_\beta \to V_\alpha \times \widetilde{V}_\beta \subset \mathbb{R}^{m+n}$$

with

$$\psi_{\alpha,\beta}(x, y) = (\varphi_\alpha(x), \widetilde{\varphi}_\beta(y)).$$

Clearly,

$$\cup_{(\alpha,\beta)\in A\times\widetilde{A}}(U_\alpha \times \widetilde{U}_\beta) = \cup_{\alpha\in A}(U_\alpha \times \cup_{\beta\in\widetilde{A}}\widetilde{U}_\beta) = \cup_{\alpha\in A}(U_\alpha \times N = (\cup_{\alpha\in A}U_\alpha) \times N) = M \times N.$$

The transition maps are

$$\psi^{-1}_{\gamma,\delta} \circ \psi_{\alpha,\beta}(x, y) = (\varphi^{-1}_\gamma \circ \varphi_\alpha(x), \widetilde{\varphi}^{-1}_\delta \circ \widetilde{\varphi}_\beta(y)),$$

which are obviously differentiable.

Finally, we have to check the Hausdorff property. Let $(x, y) \neq (z, w)$. This means that either $x \neq z$ or $y \neq w$. Choose open neighbourhoods U_x, U_z of $x, z \in M$ which either coincide (if $x = z$) or do not intersect (if $x \neq z$). Similarly, choose open neighbourhoods $\widetilde{U}_y, \widetilde{U}_w$ of $y, w \in N$ with either $\widetilde{U}_y = \widetilde{U}_w$ (if $y = w$) or $\widetilde{U}_y \cap \widetilde{U}_w = \emptyset$ (if $y \neq w$). Then $U_x \times \widetilde{U}_y \subset M \times N$ and $U_z \times \widetilde{U}_w \subset M \times N$ are open neighbourhoods of (x, y) and (z, w), respectively, and

$$(U_x \times \widetilde{U}_y) \cap (U_z \times \widetilde{U}_w) = \emptyset.$$

1.2. Consider the *Lemniscate of Gerono* Γ, which is given as a subset of $\mathbb{R}^2$ by

$$\Gamma = \{(x, y) \in \mathbb{R}^2 \mid x^4 - x^2 + y^2 = 0\}$$

We define open sets in Γ as intersections of Γ with open subsets of $\mathbb{R}^2$. Show that Γ does not admit a structure of a smooth 1-manifold.

Solution:

Observe that Γ looks like a "Figure 8". Clearly, there is a special point $p = (0, 0)$ on Γ which doesn't have a neighborhood homeomorphic to an open ball in $\mathbb{R}$ (in other words, to an open interval).

To make this intuition a bit more precise, we can see that a sufficiently small open neighborhood of p looks like a letter "X". Removing the special point p from "X" splits "X" into four pieces. But removing *any* point from an open interval makes the open interval split into *two* pieces. Observing that the number of pieces is preserved under a homeomorphism, this argument shows that no open neighbourhood of p is homeomorphic to an open interval.

To be even more precise a "piece" above is path-connected component of "X"$\backslash p$, i.e. a maximal subset of "X"$\backslash p$ such that any two of its points can be connected by a curve lying entirely in "X"$\backslash p$.

1.3. Stereographic projection

Let M be the unit 2-dimensional sphere in $\mathbb{R}^3$, $N, S \in M$, where $N = (0, 0, 1)$ and $S = (0, 0, -1)$. Define $U_N = M \setminus \{N\}$, $U_S = M \setminus \{S\}$, $V_N = V_S = \mathbb{R}^2$. Define also the map $\varphi_N : U_N \to V_N$ in the following way: if $p \in U_N$, the image $\varphi_N(p)$ is the intersection of the line through N and p with the plane $\{z = 0\}$. The map $\varphi_S : U_S \to V_S$ is defined in the same way (substitute N by S everywhere).

Compute explicitely the maps φ_N, φ_S and the transition map $\varphi_N \circ \varphi_S^{-1}$. Show that the collection $(U_\alpha, V_\alpha, \varphi_\alpha)_{\alpha\in\{S,N\}}$ is a smooth atlas, and M is a smooth manifold.

Solution:

The map φ_N can be easily computed: by construction, the point $\varphi_N(x, y, z)$ has coordinates $\lambda(x, y, z) \cdot (x, y)$, where $\lambda(x, y, z)$ is some positive number depending on the point (x, y, z). Taking a section of the sphere by a vertical plane through N and (x, y, z), one can use similarity of triangles to see that $\lambda(x, y, z) = 1/(1 - z)$, so

$$\varphi_N(x, y, z) = \frac{1}{1 - z}(x, y)$$

Similarly,

$$\varphi_S(x, y, z) = \frac{1}{1 + z}(x, y)$$

Both maps φ_N and φ_S are clearly smooth and bijective, so let us compute φ_S^{-1}. We are looking for the intersection of the line L through $(0, 0, -1)$ and $(x, y, 0)$, $(x, y) \in \mathbb{R}^2$, with the unit sphere M. This line can be parametrized by

$$\boldsymbol{L}(t) = (0, 0, -1) + t\left((x, y, 0) - (0, 0, -1)\right) = (tx, ty, -1 + t), \qquad t \in \mathbb{R}.$$

Now find $t \in \mathbb{R}$ such that $\boldsymbol{L}(t) \in M$, i.e., that

$$(tx)^2 + (ty)^2 + (-1 + t)^2 = 1.$$

This equation is equivalent to

$$t\left(t(x^2 + y^2 + 1) - 2\right) = 0,$$

hence

$$t = 0 \qquad \text{or} \qquad t = \frac{2}{x^2 + y^2 + 1}.$$

The former solution gives $\boldsymbol{L}(0) = (0, 0, -1)$, and we reject this point. We are looking for the other solution on the sphere, namely

$$\boldsymbol{L}(t) = \left(\frac{2x}{x^2+y^2+1}, \frac{2y}{-x^2-y^2+1}, -1 + \frac{2}{x^2+y^2+1}\right) = \frac{1}{x^2+y^2+1}\left(2x, 2y, -x^2-y^2+1\right).$$

Therefore,

$$\varphi_S^{-1}(x,y) = \frac{1}{x^2+y^2+1}\left(2x, 2y, -x^2-y^2+1\right).$$

We see that φ_S^{-1} is smooth, so φ_S is a diffeomorphism. Similarly, φ_N is also a diffeomorphism. Further, the image $\varphi_N(U_\alpha \cap U_\beta)$ is the plane $\mathbb{R}^2$ without the origin, and thus it is open.

Now we can compute the transition map $\varphi_N \circ \varphi_S^{-1}$. According to the formulae we have already computed,

$$\varphi_N \circ \varphi_S^{-1}(x,y) = \varphi_N\left(\frac{1}{x^2+y^2+1}\left(2x, 2y, -x^2-y^2+1\right)\right) =$$

$$= \frac{1}{1 - \frac{-x^2-y^2+1}{x^2+y^2+1}} \frac{1}{x^2+y^2+1}(2x, 2y) = \frac{1+x^2+y^2}{2x^2+2y^2}\frac{1}{x^2+y^2+1}(2x, 2y) = \frac{1}{x^2+y^2}(x, y)$$

The transition map is smooth in the domain $\mathbb{R}^2 \setminus (0,0)$. Similarly, the transition map $\varphi_S \circ \varphi_N^{-1}$ is also smooth.

We are left to verify the Hausdorff property which is straightforward.

1.4. Introduce a structure of a smooth manifold on

(a) a 2-dimensional torus $\mathbb{T}^2$ obtained from a square $[0,1] \times [0,1]$ by identification of the boundary:

$$(0,y) \sim (1,y), \quad (x,0) \sim (x,1) \qquad \forall x, y \in [0,1];$$

(b) a Klein bottle obtained from a square $[0,1] \times [0,1]$ by identification of the boundary:

$$(0,y) \sim (1,y), \quad (x,0) \sim (1-x,1) \qquad \forall x, y \in [0,1];$$

(c) a 3-dimensional torus $\mathbb{T}^3$ obtained from a cube $[0,1] \times [0,1] \times [0,1]$ by identification of the boundary:

$$(0,y,z) \sim (1,y,z), \quad (x,0,z) \sim (x,1,z), \quad (x,y,0) \sim (x,y,1) \qquad \forall x, y, z \in [0,1].$$

Solution:

We need to find a suitable atlas for every of these sets. Let us first consider the 2-torus. Consider the square $[-1,1] \times [-1,1]$ subdivided into four unit squares with horizontal and vertical sides. Introducing an equivalence relation $(x, y+1) \sim (x,y) \sim (x+1, y)$ on this square we see that it "wraps" $\mathbb{T}^2$ four times. This gives rise to an atlas consisting, say, of five charts: let V_i be unit squares centered at $(0,0)$ and $(\pm 1/2, \pm 1/2)$, and φ^{-1} be defined by the equivalence relation above: $\varphi^{-1}(x,y)$ is the (unique) point in the square $[0,1) \times [0,1)$ equivalent to (x,y). The transition maps in this atlas are translations, all the other requirements are evident.

For the Klein bottle and the 3-torus the procedure is exactly the same: for the 3-torus take eight cubes with exactly the same equivalence (the atlas will consist of nine charts), and for the Klein bottle one can use the same five charts as for torus, but the equivalence relation (in the vertical direction) should be a bit different.