Durham University
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Riemannian Geometry IV, Solutions 10 (Week 10)

10.1. (**Remark 5.19**)
Let (M, g) be a Riemannian manifold, $p \in M$, $v \in T_pM$.

(a) Show that a curve $c(t) = \exp_p(tv)$ is a geodesic.
(b) Show that every geodesic γ through p can be written as $\gamma(t) = \exp_p(tw)$ for appropriate $w \in T_pM$.

Solution: The first statement follows from the definition of the exponential map. The second one follows from the existence and uniqueness of a geodesic through p with a given tangent vector.

10.2. (**Lemma 5.20**)
Let (M, g) be a Riemannian manifold and $p \in M$. Let $\varepsilon > 0$ be small enough such that

$$\exp_p : B_\varepsilon(0_p) \to B_\varepsilon(p) \subset M$$

is a diffeomorphism. Let $\gamma : [0, 1] \to B_\varepsilon(p) \setminus \{p\}$ be any curve.

Show that there exists a curve $v : [0, 1] \to T_pM$, $\|v(s)\| = 1$ for all $s \in [0, 1]$, and a non-negative function $r : [0, 1] \to \mathbb{R}_{\geq 0}$, such that

$$\gamma(s) = \exp_p(r(s)v(s)).$$

Solution: Since the image of γ belongs to $B_\varepsilon(p)$, for every $s \in [0, 1]$ we have $\gamma(s) = \exp_p(w(s))$ for an appropriate $w(s) \in T_pM$. Then denote $r(s) = \|w(s)\|$, $v(s) = w(s)/r(s)$.

10.3. (**Lemma 5.14**)
Use the exponential map to show that any vector field $X \in \mathfrak{X}_c(M)$ along a smooth curve $c(t) : [a, b] \to M$ is a variational vector field of some variation $F(s, t)$ (i.e., $X(t) = \frac{\partial F}{\partial s}(0, t)$). Show that if $X(a) = X(b) = 0$ then the variation $F(s, t)$ can be chosen to be proper.

Solution: Assume for simplicity that for every $t \in [a, b]$ the exponential map at $c(t)$ is defined in a rather large ball. Consider the map $F(s, t) : (-\varepsilon, \varepsilon) \times [a, b] \to M$, $F(s, t) = \exp_{c(t)} sX(t)$ (check that this map is smooth!). Since $F(0, t) = \exp_{c(t)} 0 = c(t)$, this is a variation of c. Further, the variational vector field of F can be easily computed by

$$\frac{\partial F}{\partial s}(0, t) = \frac{d}{ds}(\exp_{c(t)} sX(t))\big|_{s=0} = X(t)$$

Finally, if $X(a) = X(b) = 0$, then $F(s, a) = \exp_{c(a)} 0 \equiv c(a)$ and, similarly, $F(s, b) \equiv c(b)$, so the variation is proper.

10.4. Geodesic normal coordinates
Let (M, g) be a Riemannian manifold and $p \in M$. Let $\varepsilon > 0$ such that

$$\exp_p : B_\varepsilon(0_p) \to B_\varepsilon(p) \subset M$$

is a diffeomorphism. Let $v_1, \ldots, v_n$ be an orthonormal basis of T_pM. Consider a local coordinate chart of M given by $\varphi = (x_1, \ldots, x_n) : B_\varepsilon(p) \to V = \{w \in \mathbb{R}^n \mid \|w\| < \varepsilon\}$ via

$$\varphi^{-1}(x_1, \ldots, x_n) = \exp_p(\sum_{i=1}^n x_i v_i).$$

The coordinate functions $x_1, \ldots, x_n$ of φ are called *geodesic normal coordinates.*

(a) Let g_{ij} be the metric in terms of the above coordinate system φ. Show that at the point p

$$g_{ij}(p) = \delta_{ij} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

(b) Let $w = (w_1, \dots, w_n) \in \mathbb{R}^n$ be an arbitrary vector, and $c(t) = \varphi^{-1}(tw)$. Explain why $c(t)$ is a geodesic and deduce from this fact that

$$\sum_{i,j} w_i w_j \Gamma_{ij}^k(c(t)) = 0$$

for all $1 \leq k \leq n$.

(c) Derive from (b) that all Christoffel symbols Γ_{ij}^k of the chart φ vanish at the point p (by choosing appropriate vectors $w \in \mathbb{R}^n$).

Solution:

(a) We will show that

$$\frac{\partial}{\partial x_i}\Big|_p = v_i.$$

This will imply

$$g_{ij}(p) = \langle \frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j} \rangle_p = \langle v_i, v_j \rangle_p = \delta_{ij}.$$

Denote by $\{e_i\}$ orthonormal basis in $V \subset \mathbb{R}^n$. Now, as $\varphi(p) = 0$, we can write

$$\frac{\partial}{\partial x_i}\Big|_p = \frac{d}{dt}\Big|_{t=0} \varphi^{-1}(0 + te_i) = \frac{d}{dt}\Big|_{t=0} \exp_p(tv_i) = v_i,$$

which proves (a).

(b) We have

$$c(t) = \varphi^{-1}(tw_1, \dots, tw_n) = \exp_p(t \sum_j w_j v_j).$$

Let $v = \sum_j w_j v_j \in T_pM$. Then the expression above shows that c is a geodesic with initial vector v. Let $(c_1, \dots, c_n)|_t = \varphi(c(t))$, i.e., $c_j(t) = tw_j$, $c_j'(t) = w_j$ and $c_j''(t) = 0$. Let $\frac{D}{dt}$ denote covariant derivative along c. Since c is a geodesic, we have

$$0 = \frac{D}{dt} c' = \frac{D}{dt} \sum_j c_j' \left(\frac{\partial}{\partial x_j}(c(t)) \right) = \sum_j w_j \nabla_{c'} \frac{\partial}{\partial x_j} =$$

$$= \sum_{i,j} w_i w_j \left(\nabla_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_j} \right)(c(t)) = \sum_k \left(\sum_{i,j} w_i w_j (\Gamma_{ij}^k(c(t))) \right) \frac{\partial}{\partial x_k}(c(t)).$$

Using the fact that $\frac{\partial}{\partial x_k}$ form a basis, we conclude that

$$(*) \qquad \sum_{i,j} w_i w_j \Gamma_{ij}^k(c(t)) = 0$$

for all $k \in \{1, \dots, n\}$.

(c) Evaluating $(*)$ at $t = 0$, we obtain

$$\sum_{i,j} w_i w_j \Gamma_{ij}^k(p) = 0 \quad \text{for all } w \in \mathbb{R}^n.$$

The choice $w = e_i$ yields

$$\Gamma_{ii}^k(p) = 0,$$

and then the choice $w = e_i + e_j$ yields

$$2\Gamma_{ij}^k(p) = 0,$$

so we conclude that all Christoffel symbols vanish at p. Consequently, we have

$$\nabla_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_j}(p) = 0.$$