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Michaelmas 2019

Riemannian Geometry IV, Solution 3 (Week 3)

3.1. Let M be a differentiable manifold, $U_1, U_2 \subset M$ open and $\varphi = (x_1, \ldots, x_n) : U_1 \to V_1 \subset \mathbb{R}^n$, $\psi = (y_1, \ldots, y_n) : U_2 \to V_2 \subset \mathbb{R}^n$ are two coordinate charts. Show for $p \in U_1 \cap U_2$:

$$\frac{\partial}{\partial x_i}\Big|_p = \sum_{j=1}^{n} \frac{\partial (y_j \circ \varphi^{-1})}{\partial x_i}(\varphi(p)) \cdot \frac{\partial}{\partial y_j}\Big|_p,$$

where $y_j \circ \varphi^{-1} : V_1 \to \mathbb{R}$ and $\frac{\partial (y_j \circ \varphi^{-1})}{\partial x_i}$ is the classical partial derivative in the coordinate direction x_i of $\mathbb{R}^n$.

Hint: Write $f \circ \varphi^{-1}$ as $f \circ \psi^{-1} \circ \psi \circ \varphi^{-1}$ and apply the chain rule.

Solution:

We need to check that for each function $f \in C^\infty(M, p)$ the derivation $\frac{\partial}{\partial x_i}\Big|_p$ acts in the same way as the derivation $\sum\limits_{j=1}^{n} \frac{\partial y_j}{\partial x_i} \frac{\partial}{\partial y_j}\Big|_p$.

We have

$$\frac{\partial}{\partial x_i}\Big|_p (f) = \frac{\partial (f \circ \varphi^{-1})}{\partial x_i}(\varphi(p)) = \frac{\partial}{\partial x_i}(f \circ \psi^{-1} \circ \psi \circ \varphi^{-1})(\varphi(p)).$$

The latter expression above is the partial derivative in coordinate direction x_i of the composition of the two functions $\psi \circ \varphi^{-1} : V_1 \subset \mathbb{R}^n \to V_2 \subset \mathbb{R}^n$ and $f \circ \psi^{-1} : V_2 \subset \mathbb{R}^n \to \mathbb{R}$. The chain rule tells us that

$$\frac{\partial}{\partial x_i}(f \circ \psi^{-1} \circ \psi \circ \varphi^{-1})(\varphi(p)) = \sum_{j=1}^{n} \frac{\partial (f \circ \psi^{-1})}{\partial y_j}(\psi(p)) \cdot \frac{\partial (y_j \circ \varphi^{-1})}{\partial x_i}(\varphi(p)).$$

Here $\frac{\partial}{\partial y_j}$ denotes the partial derivative in the j-th coordinate direction of $V_2 \subset \mathbb{R}^n$, and y_j in the expression $y_j \circ \varphi^{-1}$ denotes the j-th component function of the map ψ. So we finally end up with

$$\frac{\partial}{\partial x_i}\Big|_p (f) = \sum_{j=1}^{n} \frac{\partial (y_j \circ \varphi^{-1})}{\partial x_i}(\varphi(p)) \cdot \frac{\partial}{\partial y_j}|_p (f).$$

3.2. ($\star$) Let $S^2 = \{x \in \mathbb{R}^3 \mid \|x\| = 1\}$ be the standard two-dimensional sphere, let $\mathbb{R}P^2$ be the real projective plane and $\pi : S^2 \to \mathbb{R}P^2$ be the canonical projection identifying opposite points of the sphere. Let

$$c : (-\varepsilon, \varepsilon) \to S^2, \quad c(t) = (\cos t \cos(2t), \cos t \sin(2t), \sin t)$$

and

$$f : \mathbb{R}P^2 \to \mathbb{R}, \quad f(\mathbb{R}(z_1, z_2, z_3)) = \frac{(z_1 + z_2 + z_3)^2}{z_1^2 + z_2^2 + z_3^2}.$$

(a) Let $\gamma = \pi \circ c$. Calculate $\gamma'(0)(f)$.

(b) Let (φ, U) be the following coordinate chart of $\mathbb{R}P^2$:
$U = \{\mathbb{R}(z_1, z_2, z_3) \mid z_1 \neq 0\} \subset \mathbb{R}P^2$ and

$$\varphi : U \to \mathbb{R}^2, \quad \varphi(\mathbb{R}(z_1, z_2, z_3)) = \left(\frac{z_2}{z_1}, \frac{z_3}{z_1}\right).$$

Let $\varphi = (x_1, x_2)$. Express $\gamma'(t)$ in the form

$$\alpha_1(t) \frac{\partial}{\partial x_1}\Big|_{\gamma(t)} + \alpha_2(t) \frac{\partial}{\partial x_2}\Big|_{\gamma(t)}.$$

Solution:

We have $\gamma(t) = \mathbb{R} \cdot (\cos t \cos(2t), \cos t \sin(2t), \sin t)$.

(a) Since

$$(\cos t \cos(2t))^2 + (\cos t \sin(2t))^2 + (\sin t)^2 = 1,$$

we obtain

$$\gamma'(0)(f) = (f \circ \gamma)'(0) = \frac{d}{dt}\Big|_{t=0} (\cos t \cos(2t) + \cos t \sin(2t) + \sin t)^2 = 2 \cdot 3 = 6.$$

(b) Let $(\gamma_1(t), \gamma_2(t)) = \varphi \circ \gamma(t)$. Then

$$\gamma_1(t) = \tan(2t) \quad \text{and} \quad \gamma_2(t) = \frac{\tan t}{\cos(2t)}.$$

This implies that

$$\gamma'(t) = \gamma_1'(t) \frac{\partial}{\partial x_1}\Big|_{\gamma(t)} + \gamma_2'(t) \frac{\partial}{\partial x_2}\Big|_{\gamma(t)} =$$
$$= 2(1 + \tan^2(2t)) \frac{\partial}{\partial x_1}\Big|_{\gamma(t)} + \frac{(1 + \tan^2 t)\cos(2t) + 2 \tan t \sin(2t)}{\cos^2(2t)} \frac{\partial}{\partial x_2}\Big|_{\gamma(t)}.$$

3.3. The 3-sphere S^3 sits inside 2-dimensional complex space as

$$S^3 = \{(w, z) \in \mathbb{C}^2 \ : \ |w|^2 + |z|^2 = 1\}$$

(a) Writing $w = a + ib$ and $z = c + id$ we can identify the tangent space to $\mathbb{C}^2 = \mathbb{R}^4$ at the point $(1, 0) \in \mathbb{C}^2$ with the span of $\partial/\partial a, \partial/\partial b, \partial/\partial c$ and $\partial/\partial d$.
In terms of this basis, what is the subspace tangent to S^3 at $(1, 0)$?

(b) The map $\pi : S^3 \to \mathbb{C}$ given by $\pi(w, z) = z/w$ is defined away from $w = 0$. Identify the kernel of

$$D\pi : T_{(1,0)} S^3 \to T_0 \mathbb{C}.$$

Solution:

(a) If we write $|w|^2 + |z|^2 = F(w,z) = F(a,b,c,d) = a^2 + b^2 + c^2 + d^2$ then $S^3 = F^{-1}(1)$, the preimage of a regular value of F. Since F is constant along S^3, we have $DF(p)v = 0$ for any $p \in S^3$ and $v \in T_pS^3$.

Now, $DF(1,0) = (2a, 2b, 2c, 2d)|_{a=1,b=c=d=0} = (2,0,0,0)$, and this is zero on a 3-dimensional subspace which must coincide with the 3-dimensional space $T_{(1,0)}S^3$:

$$T_{(1,0)}S^3 = \left\langle \frac{\partial}{\partial b}, \frac{\partial}{\partial c}, \frac{\partial}{\partial d} \right\rangle.$$

(b) Let us write the coordinates on $\mathbb{C}$ as $\alpha + i\beta$

For the basis vectors $\frac{\partial}{\partial b}, \frac{\partial}{\partial c}, \frac{\partial}{\partial d}$ of $T_{(1,0)}S^3$ we consider some curves γ_b, γ_c and γ_d such that the directional derivatives along these curves coincide with $\frac{\partial}{\partial b}, \frac{\partial}{\partial c}, \frac{\partial}{\partial d}$. Then we consider the images of these curves under the map π and write the directional derivatives along $\pi(\gamma_b)$, $\pi(\gamma_c)$ and $\pi(\gamma_d)$ in the basis $\langle \frac{\partial}{\partial \alpha}, \frac{\partial}{\partial \beta} \rangle$.

Consider $\gamma_d(t) = (1, it)$ be a path through $(1,0) \in \mathbb{C}^2$. Then $\gamma_d'(0) = \partial/\partial d$. Now $D\pi_{(1,0)}(\gamma_d'(0)) = (\pi \circ \gamma_d)'(0)$ and $(\pi \circ \gamma_d)(t) = \frac{it}{1} = it$ so we see that

$$D\pi_{(1,0)}\left(\frac{\partial}{\partial d}\right) = \frac{\partial}{\partial \beta} \in T_0\mathbb{C}$$

Similarly we choose $\gamma_c(t) = (1,t)$ and see that $(\pi \circ \gamma_c)(t) = \frac{t}{1} = t$, so

$$D\pi_{(1,0)}\left(\frac{\partial}{\partial c}\right) = \frac{\partial}{\partial \alpha} \in T_0\mathbb{C}.$$

Finally, take $\gamma_b(t) = (1 + it, 0)$, so that $(\pi \circ \gamma_b)(t) = \frac{0}{1+it} = 0$ and

$$D\pi_{(1,0)}\left(\frac{\partial}{\partial b}\right) = 0 \in T_0\mathbb{C}$$

Hence we see that the kernel of $D\pi_{(1,0)}$ is just the 1-dimensional vector space spanned by $\frac{\partial}{\partial b}$.

3.4. ($\star$) Show that the tangent space of the Lie group $SO_n(\mathbb{R}) \subset M_n(\mathbb{R}) \cong \mathbb{R}^{n^2}$ (see Exercise 2.3) at the identity $I \in SO_n(\mathbb{R})$ is given by

$$T_I\, SO_n(\mathbb{R}) = \{A \in M_n(\mathbb{R}) \mid A^t = -A\},$$

i.e., the space of all skew-symmetric $n \times n$-matrices.

Hint: You may use that we have, componentwise, $(AB)'(s) = A'(s)B(s) + A(s)B'(s)$ for the product of any two matrix-valued curves, and $(A^t)'(s) = (A'(s))^t$.

Solution:

Let $A : (-\varepsilon, \varepsilon) \to SO_n(\mathbb{R})$ be a smooth curve on the smooth manifold $SO_n(\mathbb{R})$ with $A(0) = I$. Then we know that

$$A(s)(A(s))^t = I,$$

for all $s \in (-\varepsilon, \varepsilon)$. Differentiation gives

$$A'(0)(A(0))^t + A(0)(A'(0))^t = A'(0)I^t + I(A'(0))^t = A'(0) + (A'(0))^t = 0.$$

So we conclude that

$$T_I SO(n) \subset \{B \in M_n(\mathbb{R}) \mid B + B^t = 0\}.$$

The right hand side is the space of all skew-symmetric $n \times n$-matrices, which is a vector space of dimension $\frac{n(n-1)}{2}$. Since $SO_n(\mathbb{R})$ is a differentiable manifold of dimension $\frac{n(n-1)}{2}$, its tangent space $T_I SO_n(\mathbb{R})$ is a vector space of the same dimension. Since both vector spaces have the same dimension, the above inclusion is actually an equality, i.e.,

$$T_I SO_n(\mathbb{R}) = \{B \in M_n(\mathbb{R}) \mid B + B^t = 0\}.$$