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Riemannian Geometry IV, Homework 3 (Week 3)

Due date for starred problems: **Friday, November 8**.

3.1. Let M be a differentiable manifold, $U_1, U_2 \subset M$ open and $\varphi = (x_1, \dots, x_n) : U_1 \to V_1 \subset \mathbb{R}^n$, $\psi = (y_1, \dots, y_n) : U_2 \to V_2 \subset \mathbb{R}^n$ are two coordinate charts. Show for $p \in U_1 \cap U_2$:

$$\frac{\partial}{\partial x_i}\Big|_p = \sum_{j=1}^n \frac{\partial (y_j \circ \varphi^{-1})}{\partial x_i}(\varphi(p)) \cdot \frac{\partial}{\partial y_j}\Big|_p,$$

where $y_j \circ \varphi^{-1} : V_1 \to \mathbb{R}$ and $\frac{\partial (y_j \circ \varphi^{-1})}{\partial x_i}$ is the classical partial derivative in the coordinate direction x_i of $\mathbb{R}^n$.

Hint: Write $f \circ \varphi^{-1}$ as $f \circ \psi^{-1} \circ \psi \circ \varphi^{-1}$ and apply the chain rule.

3.2. ($\star$) Let $S^2 = \{x \in \mathbb{R}^3 \mid \|x\| = 1\}$ be the standard two-dimensional sphere, let $\mathbb{R}P^2$ be the real projective plane and $\pi : S^2 \to \mathbb{R}P^2$ be the canonical projection identifying opposite points of the sphere. Let

$$c : (-\varepsilon, \varepsilon) \to S^2, \quad c(t) = (\cos t \cos(2t), \cos t \sin(2t), \sin t)$$

and

$$f : \mathbb{R}P^2 \to \mathbb{R}, \quad f(\mathbb{R}(z_1, z_2, z_3)) = \frac{(z_1 + z_2 + z_3)^2}{z_1^2 + z_2^2 + z_3^2}.$$

(a) Let $\gamma = \pi \circ c$. Calculate $\gamma'(0)(f)$.

(b) Let (φ, U) be the following coordinate chart of $\mathbb{R}P^2$:
$U = \{\mathbb{R}(z_1, z_2, z_3) \mid z_1 \neq 0\} \subset \mathbb{R}P^2$ and

$$\varphi : U \to \mathbb{R}^2, \quad \varphi(\mathbb{R}(z_1, z_2, z_3)) = \left(\frac{z_2}{z_1}, \frac{z_3}{z_1}\right).$$

Let $\varphi = (x_1, x_2)$. Express $\gamma'(t)$ in the form

$$\alpha_1(t) \frac{\partial}{\partial x_1}\Big|_{\gamma(t)} + \alpha_2(t) \frac{\partial}{\partial x_2}\Big|_{\gamma(t)}.$$

3.3. The 3-sphere S^3 sits inside 2-dimensional complex space as

$$S^3 = \{(w, z) \in \mathbb{C}^2 \ : \ |w|^2 + |z|^2 = 1\}$$

(a) Writing $w = a + ib$ and $z = c + id$ we can identify the tangent space to $\mathbb{C}^2 = \mathbb{R}^4$ at the point $(1, 0) \in \mathbb{C}^2$ with the span of $\partial/\partial a, \partial/\partial b, \partial/\partial c$ and $\partial/\partial d$.
In terms of this basis, what is the subspace tangent to S^3 at $(1, 0)$?

(b) The map $\pi : S^3 \to \mathbb{C}$ given by $\pi(w, z) = z/w$ is defined away from $w = 0$. Identify the kernel of

$$D\pi : T_{(1,0)} S^3 \to T_0 \mathbb{C}.$$

3.4. ($\star$) Show that the tangent space of the Lie group $SO_n(\mathbb{R}) \subset M_n(\mathbb{R}) \cong \mathbb{R}^{n^2}$ (see Exercise 2.3) at the identity $I \in SO_n(\mathbb{R})$ is given by

$$T_I \, SO_n(\mathbb{R}) = \{A \in M_n(\mathbb{R}) \mid A^t = -A\},$$

i.e., the space of all skew-symmetric $n \times n$-matrices.

Hint: You may use that we have, componentwise, $(AB)'(s) = A'(s)B(s) + A(s)B'(s)$ for the product of any two matrix-valued curves, and $(A^t)'(s) = (A'(s))^t$.