Riemannian Geometry IV, Solutions 4 (Week 4)

4.1. Let M and N be smooth manifolds. Using local coordinates, explain why $T_{(p,q)}(M \times N) = T_p M \oplus T_q N$ for $p \in M$ and $q \in N$.

Solution:

If (U_i, V_i, φ_i) , $i \in I$, is an atlas for M, and (U_j, V_j, ψ_j) , $j \in J$, is an atlas for N, then we get an atlas for $M \times N$ by taking products $(U_i \times U_j, V_i \times V_j, \varphi_i \times \psi_j)$ for $(i, j) \in I \times J$. It follows that if (x_1, \ldots, x_m) and (y_1, \ldots, y_n) are local coordinates at $p \in M$ and $q \in N$, then $(x_1, \ldots, x_m, y_1, \ldots, y_n)$ are local coordinates at $(p, q) \in M \times N$. Hence we see that

$$\begin{split} T_{(p,q)}(M\times N) &= \langle \frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_m}, \frac{\partial}{\partial y_1}, \dots, \frac{\partial}{\partial y_n} \rangle \\ &= \langle \frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_m} \rangle \oplus \langle \frac{\partial}{\partial y_1}, \dots, \frac{\partial}{\partial y_n} \rangle = T_p M \oplus T_q N. \end{split}$$

4.2. Let $M \subset \mathbb{R}^n$ be a smooth manifold given by the equation $f(x_1, \ldots, x_n) = a$. Let $p \in M$ and $v \in T_pM$. Show that the vector $v = (v_1, \ldots, v_n)$ satisfies the equation

$$\sum_{i=1}^{n} \frac{\partial f}{\partial x_i} v_i = 0,$$

or, equivalently, $\langle \operatorname{grad} f(p), v \rangle = 0$.

Solution:

Let $\gamma(t)$ be a curve on M such that $\gamma(0) = p$ and $v = \gamma'(0)$. Then we have $f(\gamma_1(t), \dots, \gamma_n(t)) = a$. We compute $\frac{df(\gamma(t))}{dt}\big|_{t=0}$ using the chain rule:

$$\frac{df(\gamma(t))}{dt}\big|_{t=0} = \sum_{i=1}^n \frac{\partial f}{\partial x_i} \frac{d\gamma_i(t)}{dt}\big|_{t=0} = \sum_{i=1}^n \frac{\partial f}{\partial x_i} v_i$$

Notice that as $f(\gamma(t)) \equiv a$, we also have $\frac{df(\gamma(t))}{dt}\big|_{t=0} = 0$ which implies

$$\sum_{i=1}^{n} \frac{\partial f}{\partial x_i} = 0.$$

4.3. (\star) Let X be a vector field on \mathbb{R}^3 defined by

$$X(x,y,z) = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y} + (x+y+z)\frac{\partial}{\partial z}$$

Let $M \subset \mathbb{R}^3$ be a cylinder $\{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 = 1\}$.

- (a) Show that $X \in \mathfrak{X}(M)$.
- (b) Express X in terms of $\frac{\partial}{\partial \varphi}$ and $\frac{\partial}{\partial h}$, where (φ, h) are cylindrical coordinates on M, i.e.

$$(x, y, z) = (\cos \varphi, \sin \varphi, h)$$

Solution:

(a) Identifying $T_{(x,y,z)}\mathbb{R}^3$ with \mathbb{R}^3 , we can write that $X(x,y,z)=(y,-x,x+y+z)\in\mathbb{R}^3$. The cylinder M is a level set of a function $f(x,y,z)=x^2+y^2$, thus, according to Exercise 4.2, vector X(x,y,z) is tangent to the cylinder if and only if $\langle \operatorname{grad} f(x,y,z), X(x,y,z) \rangle = 0$. Since $\operatorname{grad} f(x,y,z) = (2x,2y,0)$, we see that

$$\langle \text{grad } f(x, y, z), X(x, y, z) \rangle = \langle (2x, 2y, 0), (y, -x, x + y + z) \rangle = 0$$

(b) Since $(x, y, z) = (\cos \varphi, \sin \varphi, h)$, we have

$$\begin{split} \frac{\partial}{\partial \varphi} &= (-\sin \varphi, \cos \varphi, 0) = -\sin \varphi \frac{\partial}{\partial x} + \cos \varphi \frac{\partial}{\partial y}, \\ \frac{\partial}{\partial h} &= (0, 0, 1) = \frac{\partial}{\partial z}, \\ X(x, y, z) &= (\sin \varphi, -\cos \varphi, \cos \varphi + \sin \varphi + h). \end{split}$$

Thus,

$$X(\varphi,h) = -\frac{\partial}{\partial \varphi} + (\cos \varphi + \sin \varphi + h) \frac{\partial}{\partial h}$$

- **4.4.** (a) (\star) Find vector fields $X, Y \in \mathfrak{X}(\mathbb{T}^2)$ such that $\{X(p), Y(p)\}$ is a basis for $T_p\mathbb{T}^2$ for all $p \in \mathbb{T}^2$. **Hint:** you may embed the torus \mathbb{T}^2 into \mathbb{R}^3 as a surface of revolution.
 - (b) Find vector fields $X, Y, Z \in \mathfrak{X}(S^3)$ such that $\{X(p), Y(p), Z(p)\}$ is a basis for T_pS^3 for all $p \in S^3$.

Hint: you may use the embedding of S^3 described in Exercise 3.3.

Solution:

(a) We may embed the torus into \mathbb{R}^3 as

$$(x, y, z) = ((\cos \vartheta + 2)\cos \varphi, (\cos \vartheta + 2)\sin \varphi, \sin \vartheta),$$

where $\vartheta, \varphi \in [0, 2\pi)$. The curves $\vartheta = \text{const}$ are parallels of the torus, the curves $\varphi = \text{const}$ are meridians. It is easy to see that at every point of \mathbb{T}^2 the meridian is orthogonal to the parallel, so tangent vectors to them compose a basis of the tangent space. Thus, it is sufficient to find two non-vanishing vector fields X and Y, where X is tangent to meridians, and Y is tangent to parallels. We may define

$$\begin{split} X(\vartheta,\varphi) &=& \frac{\partial}{\partial \vartheta} = (-\sin\vartheta\cos\varphi, -\sin\vartheta\sin\varphi, \cos\vartheta), \\ Y(\vartheta,\varphi) &=& \frac{\partial}{\partial \varphi} = (-(\cos\vartheta + 2)\sin\varphi, (\cos\vartheta + 2)\cos\varphi, 0). \end{split}$$

The fields X and Y can also be written in terms of (x, y, z)-coordinates in \mathbb{R}^3 :

$$X(x,y,z) = \left(\frac{-xz}{\sqrt{x^2 + y^2}}, \frac{-yz}{\sqrt{x^2 + y^2}}, \sqrt{1 - z^2} \operatorname{sgn}(x^2 + y^2 - 4)\right),$$

$$Y(x,y,z) = (-y,x,0).$$

(b) Embedding S^3 as the unit sphere into \mathbb{R}^4 , we see that choosing the vector

$$(-y, x, -w, z) \in T_{(x,y,z,w)}S^3 \subset T_{(x,y,z,w)}\mathbb{R}^4$$

describes a nowhere-vanishing vector field (you should check that this vector actually lies in $T_{(x,y,z,w)}S^3$ by checking that it is normal to the normal direction to S^3 at this point).

Permuting coordinates in an appropriate way, you then should be able to find (up to an overall multiplication by ± 1) five other nowhere-vanishing vector fields. Choosing carefully three of these, you should then verify that the vectors at each point of S^3 compose a basis of the tangent space. You can do this by checking that they are linearly independent.