Durham University
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Michaelmas 2019

Riemannian Geometry IV, Homework 4 (Week 4)

Due date for starred problems: **Friday, November 8**.

4.1. Let M and N be smooth manifolds. Using local coordinates, explain why $T_{(p,q)}(M \times N) = T_pM \oplus T_qN$ for $p \in M$ and $q \in N$.

4.2. Let $M \subset \mathbb{R}^n$ be a smooth manifold given by the equation $f(x_1, \dots, x_n) = a$, where $f \in C^\infty(\mathbb{R}^n)$. Let $p \in M$ and $v \in T_pM$. Show that the vector $v = (v_1, \dots, v_n)$ satisfies the equation

$$\sum_{i=1}^{n} \frac{\partial f}{\partial x_i} v_i = 0,$$

or, equivalently, $\langle \text{grad } f(p), v \rangle = 0$.

4.3. ($\star$) Let X be a vector field on $\mathbb{R}^3$ defined by

$$X(x, y, z) = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y} + (x + y + z)\frac{\partial}{\partial z}$$

Let $M \subset \mathbb{R}^3$ be a cylinder $\{(x, y, z) \in \mathbb{R}^3 \,|\, x^2 + y^2 = 1\}$.

(a) Show that $X \in \mathfrak{X}(M)$.

(b) Express X in terms of $\frac{\partial}{\partial \varphi}$ and $\frac{\partial}{\partial h}$, where (φ, h) are cylindrical coordinates on M, i.e.

$$(x, y, z) = (\cos\varphi, \sin\varphi, h)$$

4.4. (a) ($\star$) Find vector fields $X, Y \in \mathfrak{X}(\mathbb{T}^2)$ such that $\{X(p), Y(p)\}$ is a basis for $T_p\mathbb{T}^2$ for all $p \in \mathbb{T}^2$.
Hint: you may embed the torus $\mathbb{T}^2$ into $\mathbb{R}^3$ as a surface of revolution.

(b) Find vector fields $X, Y, Z \in \mathfrak{X}(S^3)$ such that $\{X(p), Y(p), Z(p)\}$ is a basis for T_pS^3 for all $p \in S^3$.
Hint: you may use the embedding of S^3 described in Exercise 3.3.