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Riemannian Geometry IV, Solutions 5 (Week 5)

5.1. ($\star$) Let M be a smooth manifold and let $X, Y, Z \in \mathfrak{X}(M)$ be vector fields on M, and let $a \in \mathbb{R}$. Prove the following identities concerning the Lie bracket:

(a) Linearity $[X + aY, Z] = [X, Z] + a[Y, Z]$.

(b) Anti-symmetry $[Y, X] = -[X, Y]$.

(c) Jacobi identity $[[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] = 0$.

Solution:

(a) Note that $Z(ag) = aZ(g)$ for constants $a \in \mathbb{R}$ since

$$\frac{\partial}{\partial x_i}(ag) = a\frac{\partial g}{\partial x_i},$$

and the same holds for linear combinations of these basis vector fields. Thus, we have

$$\begin{aligned}[X + aY, Z]f &= (X + aY)Zf - Z(X + aY)f \\ &= XZf + aYZf - ZXf - aZYf = (XZf - ZXf) + a(YZf - ZYf) \\ &= [X, Z]f + a[Y, Z]f.\end{aligned}$$

(b) We have

$$[X, Y]f = XYf - YXf = -(YXf - XYf) = -[Y, X]f$$

for all $f \in C^\infty(M)$. This implies that $[X, Y] = -[Y, X]$.

(c) Using (b), it is enough to show that

$$[[X, Y], Z] = [X, [Y, Z]] + [Y, [Z, X]].$$

The left hand side, applied to a function $f \in C^\infty(M)$, is

$$[[X, Y], Z]f = [X, Y]Zf - Z[X, Y]f = XYZf - YXZf - ZXYf + ZYXf.$$

The right hand side, applied to the same function, is

$$\begin{aligned}&[X, [Y, Z]]f + [Y, [Z, X]]f = \\ &= XYZf - XZYf - YZXf + ZYXf + YZXf - YXZf - ZXYf + XZYf = \\ &= XYZf + ZYXf - YXZf - ZXYf,\end{aligned}$$

which is notably the same. This proves Jacobi identity.

The Hairy Ball Theorem. Let $S^n \subset \mathbb{R}^{n+1}$ denote the unit n-sphere. If n is even, then there is no continuous non-vanishing vector field $X \in \mathfrak{X}(S^n)$.

This theorem tells us for example that it can not be windy everywhere at once on Earth's surface – at any given moment, the horizontal wind speed somewhere must be zero.

Exercise 4.4(b) shows that The Hairy Ball Theorem does not hold in odd dimensions. Moreover, it can be generalized in the following way.

5.2. (a) Find a non-vanishing vector field on S^{2m+1} for arbitrary m.

(b) Construct $2m+1$ vector fields on S^{2m+1} forming a basis of T_pS^{2m+1} at every point $p \in S^{2m+1}$.

Solution:

(a) Embedding S^{2m-1} as the unit sphere inside $\mathbb{R}^{2m}$ (with coordinates $x_1, \dots, x_{2m}$), we may take the vector field given by

$$(-x_2, x_1, -x_4, x_3, \dots, -x_{2m}, x_{2m-1})$$

(cf. Exercise 4.4(b)).

(b) The solution is similar to one of Exercise 4.4(b). Permuting the coordinates of the field above, you may get plenty of nowhere-vanishing fields. Then, choosing carefully $2n-1$ linearly independent (at every point!) ones, you get required basis.

5.3. Tangent space of a matrix group as a Lie algebra

Let $G \subset M_n(\mathbb{R})$ be a matrix group and $h \in G$. We consider the tangent space T_hG as a subspace of $M_n(\mathbb{R})$.

(a) Let $g(s) \in G$ be a path in G with $g(0) = I$, and let $g'(0) = A \in T_IG \subset M_n(\mathbb{R})$. Let $\gamma(s) = g^{-1}(s)$. Show that $\gamma'(0) = -A$.

(b) Let $g \in G$ and $A \in T_IG \subset M_n(\mathbb{R})$. Show that $gAg^{-1} \in T_IG$. (The map $\mathrm{Ad}_g : T_IG \to T_IG$ sending $A \in T_IG$ to $gAg^{-1} \in T_IG$ is called an *adjoint representation* of G).

(c) Show that the tangent space T_hG at $h \in G$ can be obtained from T_IG by multiplying all the matrices from T_IG by h from the left: $T_hG = hT_IG$. Show that T_hG can also be obtained from T_IG by multiplying all the matrices from T_IG by h from the right.

(d) Show that for every $A \in T_IG$ there exists a vector field $X \in \mathfrak{X}(G)$ with $X(I) = A$.

Hint: try to find a *left-invariant field*, i.e. a field satisfying $X(gh) = gX(h)$ for $g, h \in G$.

(e) Show that if $A, B \in T_IG$, then $[A, B] = AB - BA$ is also an element of T_IG.

Remark: Exercise 5.3 can be generalized to any Lie group, we will see it in the next term.

Solution:

(a) Differentiating the equality $\gamma(s)g(s) = I$ at $s = 0$ we get $\gamma'(0) + g'(0) = 0$, which implies $\gamma'(0) = -g'(0) = -A$.

(b) Let $A = \gamma'(0)$ for some curve $\gamma(s)$ in G through I at $s = 0$. Then gAg^{-1} is the tangent vector at 0 of the curve $g\gamma(s)g^{-1}$.

(c) Let $\gamma(s)$ be a curve in G, $\gamma(0) = I$, and let $h \in G$. Then $h\gamma(s)$ is also a curve in G, however $h\gamma(0) = h$, and thus the derivative of $h\gamma(s)$ at $s = 0$ is an element of T_hG. Differentiating this curve at 0, we get

$$h\gamma'(0) = \frac{d}{ds} h\gamma(s)\Big|_{s=0} \in T_hG$$

Thus, for any $A \in T_IG$ we have $hA \in T_hG$. Since the map $A \to hA$ is clearly injective and the dimensions of T_IG and T_hG coinside, we see that $T_hG = hT_IG$. In exactly the same way we can see that $T_hG = (T_IG)h$ (note: the two maps $T_IG \to T_hG$ defined by $A \to hA$ and $A \to Ah$ are distinct).

(d) Take $A \in T_IG$. Define $X = X_A \in \mathfrak{X}(G)$ as $X(h) = hA$. According to (c), $X(h) \in T_hG$, and clearly $X(h)$ depends on h smoothly.

(e) If $A, B \in T_IG$, then $[A, B] = AB - BA = [X_A, X_B](I) \in T_IG$, where $X_A, X_B \in \mathfrak{X}(G)$ are defined in (d).

5.4. ($\star$) Let $\mathbb{H}^2$ be the upper half-plane model of hyperbolic 2-space. Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{R})$ and define the map

$$f_A : \mathbb{H}^2 \to \mathbb{H}^2, \ f_A(z) = \frac{az+b}{cz+d}.$$

(a) Show that $f_A \circ f_B = f_{AB}$.

(b) Show that for every $A \in SL_2(\mathbb{R})$ the map f_A is an isometry of $\mathbb{H}^2$.
Hint: show first that
$$\mathrm{Im}(f_A(z)) = \frac{\mathrm{Im}(z)}{|cz+d|^2}.$$

Solution:

(a) This can be done by an explicit computation: if $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, B = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \in SL_2(\mathbb{R})$, then

$$f_A \circ f_B(z) = f_A\left(\frac{a'z+b'}{c'z+d'}\right) = \frac{a\left(\frac{a'z+b'}{c'z+d'}\right)+b}{c\left(\frac{a'z+b'}{c'z+d'}\right)+d} =$$
$$= \frac{a(a'z+b')+b(c'z+d')}{c(a'z+b')+d(c'z+d')} = \frac{(aa'+bc')z+(ab'+bd')}{(ca'+dc')z+(cb'+dd')} = f_{AB}(z)$$

(b) We first follow the hint:

$$\begin{aligned} \mathrm{Im}(f_A(z)) &= \mathrm{Im}\left(\frac{az+b}{cz+d}\right) \\ &= \mathrm{Im}\left(\frac{(az+b)(c\overline{z}+d)}{|cz+d|^2}\right) \\ &= \mathrm{Im}\left(\frac{ac|z|^2+bd+adz+bc\overline{z}}{|cz+d|^2}\right) \\ &= \mathrm{Im}\left(\frac{i(ad-bc)\mathrm{Im}(z)}{|cz+d|^2}\right) \\ &= \frac{\mathrm{Im}(z)}{|cz+d|^2}. \end{aligned}$$

Now we want to show that f_A is an isometry of $\mathbb{H}^2$, in other words, that it preserves the Riemannian metric. In fact, it is enough to show that it preserves the Riemannian norm $\|\cdot\|^2 = \langle\cdot,\cdot\rangle$.
First, we need to calculate the differential of f_A. Let $z(t)$ be a curve in $\mathbb{H}^2 \subset \mathbb{C}$, $z : \mathbb{R} \to \mathbb{H}^2$, then

$$\begin{aligned} Df_A(z'(0)) &= \left.\frac{d}{dt}\right|_{t=0} \frac{az(t)+b}{cz(t)+d} \\ &= \frac{(ad-bc)z'(0)}{(cz(0)+d)^2} \\ &= \frac{z'(0)}{(cz(0)+d)^2}. \end{aligned}$$

Then we see that

$$\begin{aligned} \langle Df_A(z'(0)), Df_A(z'(0))\rangle &= \frac{1}{[\mathrm{Im}f_A(z(0))]^2}\frac{|z'(0)|^2}{|(cz(0)+d)|^4} \\ &= \frac{|z'(0)|^2}{[\mathrm{Im}z(0)]^2} \\ &= \langle z'(0), z'(0)\rangle. \end{aligned}$$

Therefore, f_A preserves the Riemannian norm, and hence it is an isometry.