Durham University
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# Riemannian Geometry IV, Homework 5 (Week 5)

**Due date** for starred problems: **Friday, November 22**.

**5.1.** ($\star$) Let $M$ be a smooth manifold and let $X, Y, Z \in \mathfrak{X}(M)$ be vector fields on $M$, and let $a \in \mathbb{R}$. Prove the following identities concerning the Lie bracket:

(a) Linearity $[X + aY, Z] = [X, Z] + a[Y, Z]$.
(b) Anti-symmetry $[Y, X] = -[X, Y]$.
(c) Jacobi identity $[[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] = 0$.

**The Hairy Ball Theorem.** Let $S^n \subset \mathbb{R}^{n+1}$ denote the unit $n$-sphere. If $n$ is even, then there is no continuous non-vanishing vector field $X \in \mathfrak{X}(S^n)$.

This theorem tells us for example that it can not be windy everywhere at once on Earth's surface – at any given moment, the horizontal wind speed somewhere must be zero.

Exercise 4.4(b) shows that The Hairy Ball Theorem does not hold in odd dimensions. Moreover, it can be generalized in the following way.

**5.2.** (a) Find a non-vanishing vector field on $S^{2m+1}$ for arbitrary $m$.
(b) Construct $2m+1$ vector fields on $S^{2m+1}$ forming a basis of $T_pS^{2m+1}$ at every point $p \in S^{2m+1}$.

**5.3. Tangent space of a matrix group as a Lie algebra**
Let $G \subset M_n(\mathbb{R})$ be a matrix group and $h \in G$. We consider the tangent space $T_hG$ as a subspace of $M_n(\mathbb{R})$.

(a) Let $g(s) \in G$ be a path in $G$ with $g(0) = I$, and let $g'(0) = A \in T_IG \subset M_n(\mathbb{R})$. Let $\gamma(s) = g^{-1}(s)$. Show that $\gamma'(0) = -A$.
(b) Let $g \in G$ and $A \in T_IG \subset M_n(\mathbb{R})$. Show that $gAg^{-1} \in T_IG$. (The map $\mathrm{Ad}_g : T_IG \to T_IG$ sending $A \in T_IG$ to $gAg^{-1} \in T_IG$ is called an *adjoint representation* of $G$).
(c) Show that the tangent space $T_hG$ at $h \in G$ can be obtained from $T_IG$ by multiplying all the matrices from $T_IG$ by $h$ from the left: $T_hG = hT_IG$. Show that $T_hG$ can also be obtained from $T_IG$ by multiplying all the matrices from $T_IG$ by $h$ from the right.
(d) Show that for every $A \in T_IG$ there exists a vector field $X \in \mathfrak{X}(G)$ with $X(I) = A$.
**Hint:** try to find a *left-invariant field*, i.e. a field satisfying $X(gh) = gX(h)$ for $g, h \in G$.
(e) Show that if $A, B \in T_IG$, then $[A, B] = AB - BA$ is also an element of $T_IG$.

*Remark:* Exercise 5.3 can be generalized to any Lie group, we will see it in the next term.

**5.4.** ($\star$) Let $\mathbb{H}^2$ be the upper half-plane model of hyperbolic 2-space. Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{R})$ and define the map

$$f_A : \mathbb{H}^2 \to \mathbb{H}^2, \ f_A(z) = \frac{az + b}{cz + d}.$$

(a) Show that $f_A \circ f_B = f_{AB}$.
(b) Show that for every $A \in SL_2(\mathbb{R})$ the map $f_A$ is an isometry of $\mathbb{H}^2$.
**Hint:** show first that

$$\mathrm{Im}(f_A(z)) = \frac{\mathrm{Im}(z)}{|cz + d|^2}.$$