Durham University
Pavel Tumarkin

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Riemannian Geometry IV, Solutions 6 (Week 6)

6.1. Let X and Y be two vector fields on $\mathbb{R}^3$ defined by

$$\begin{aligned} X(x,y,z) &= z\frac{\partial}{\partial x} - 2z\frac{\partial}{\partial y} + (2y-x)\frac{\partial}{\partial z}, \\ Y(x,y,z) &= y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}, \end{aligned}$$

and let S^2 sit inside $\mathbb{R}^3$ as the sphere of radius 1 centered at the origin.

(a) Compute the Lie bracket $[X,Y]$.

(b) Verify that the restrictions of the vector fields X and Y to S^2 are vector fields on S^2 (in other words, are everywhere tangent to S^2).

(c) Check that the restriction of $[X,Y]$ to S^2 is also a vector field on S^2.

Solution:

(a) You should get

$$[X,Y] = -2z\frac{\partial}{\partial x} - z\frac{\partial}{\partial y} + (2x+y)\frac{\partial}{\partial z}.$$

(b) The vector $X(x,y,z) \in T_{(x,y,z)}\mathbb{R}^3$ belongs to the space $T_{(x,y,z)}S^2 \subset T_{(x,y,z)}\mathbb{R}^3$ if it is orthogonal to the normal direction to S^2 at (x,y,z). The normal direction to S^2 at (x,y,z) is given by the vector $n = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}$. Taking the dot product we get

$$n \cdot X(x,y,z) = xz - 2yz + z(2y-x) = 0$$

as required. One can do similar calculations for Y and for $[X,Y]$.

6.2. ($\star$) **Isometry between the hyperboloid and unit ball models of the hyperbolic plane**

Let $\mathbb{W}^2 = \{x \in \mathbb{R}^3 \mid q(x,x) = -1,\ x_3 > 0\}$ with $q(x,y) = x_1y_1 + x_2y_2 - x_3y_3$ be the hyperboloid model of the hyperbolic plane. Let the Poincaré unit ball model $\mathbb{B}^2$ of hyperbolic 2-space sit inside $\mathbb{R}^3$ as $\mathbb{B}^2 = \{x \in \mathbb{R}^3 \mid x_3 = 0,\ x_1^2 + x_2^2 < 1\}$.

We define a map $f : \mathbb{W}^2 \to \mathbb{B}^2$ by requiring that for each $p \in \mathbb{W}^2$ the points $f(p) \in \mathbb{B}^2$ and p are collinear with the point $(0,0,-1)$ (i.e. f is a projection from this point to the plane $\{z=0\}$).

(a) Calculate explicitly the maps $f(X,Y,Z)$ for $(X,Y,Z) \in \mathbb{W}^2$ and $f^{-1}(x,y,0)$ for $(x,y,0) \in \mathbb{B}^2$. **Hint:** you will obtain

$$x = \frac{X}{Z+1}, \quad y = \frac{Y}{Z+1}.$$

and

$$f^{-1}(x,y) = \left(\frac{2x}{1-x^2-y^2}, \frac{2y}{1-x^2-y^2}, \frac{1+x^2+y^2}{1-x^2-y^2}\right).$$

(b) An almost global coordinate chart $\varphi : U \to V$ on $\mathbb{W}^2$ is given by

$$\varphi^{-1}(x_1,x_2) = (\cos(x_1)\sinh(x_2), \sin(x_1)\sinh(x_2), \cosh(x_2)),$$

where $0 < x_1 < 2\pi$ and $0 < x_2 < \infty$. Let $\psi = \varphi \circ f^{-1}$ be a coordinate chart on $\mathbb{B}^2$ with coordinate functions y_1, y_2. Calculate ψ^{-1} explicitly.

(c) Explain why

$$Df(p)(\frac{\partial}{\partial x_i}) = \frac{\partial}{\partial y_i}$$

for for all $p \in U$ and $i = 1, 2$, where $\frac{\partial}{\partial x_i} \in T_p\mathbb{W}^2$ and $\frac{\partial}{\partial y_i} \in T_{f(p)}\mathbb{B}^2$.

(d) Show that

$$\langle \frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j} \rangle_p = \langle \frac{\partial}{\partial y_i}, \frac{\partial}{\partial y_j} \rangle_{f(p)}$$

for all $p \in U$, and $i, j \in \{1, 2\}$. Together with part (c), this demonstrates that f is an isometry.

Additional remark. To be precise, we need to choose two coordinate charts of the above type with $V_1 = (0, 2\pi) \times (0, \infty)$ and $V_2 = (-\pi, \pi) \times (0, \infty)$, and to consider also the linear map $Df(0, 0, 1) : T_{(0,0,1)}\mathbb{W}^2 \to T_0\mathbb{B}^2$ to cover the whole hyperbolic plane and to fully prove that f is an isometry.

Solution:

(a) Write $f(X, Y, Z) = (x, y)$. Then since $(x, y, 0)$ lies on the line containing (X, Y, Z) and $(0, 0, -1)$, we have $x/X = y/Y = 1/(Z + 1)$ so that

$$x = \frac{X}{Z+1}, \quad y = \frac{Y}{Z+1}.$$

Similarly, one can show that

$$f^{-1}(x, y) = \left(\frac{2x}{1 - x^2 - y^2}, \frac{2y}{1 - x^2 - y^2}, \frac{1 + x^2 + y^2}{1 - x^2 - y^2} \right).$$

(b) Since $\psi = \varphi \circ f^{-1}$, we have $\psi^{-1} = f \circ \varphi^{-1}$, and we know both f and φ^{-1} explicitly. So we see that

$$\psi^{-1}(y_1, y_2) = \left(\frac{\sinh y_2 \cos y_1}{1 + \cosh y_2}, \frac{\sinh y_2 \sin y_1}{1 + \cosh y_2} \right).$$

(c) By the definition of ψ, the map $\psi \circ f \circ \varphi^{-1}$ between the two charts is just the identity map. So, $Df(p)(\frac{\partial}{\partial x_i}) = \frac{\partial}{\partial y_i}$.

(d) We compute:

$$\frac{\partial}{\partial x_1} = (-\sin x_1 \sinh x_2, \cos x_1 \sinh x_2, 0),$$

$$\frac{\partial}{\partial x_2} = (\cos x_1 \cosh x_2, \sin x_1 \cosh x_2, \sinh x_2).$$

Now, using the metric defined via the form q on the hyperboloid, we see that

$$\left\langle \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_1} \right\rangle = \sinh^2 x_2,$$

$$\left\langle \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2} \right\rangle = 0,$$

$$\left\langle \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_2} \right\rangle = \cosh^2 x_2 - \sinh^2 x_2 = 1.$$

We also compute:

$$\frac{\partial}{\partial y_1} = \left(\frac{-\sinh y_2 \sin y_1}{1 + \cosh y_2}, \frac{\sinh y_2 \cos y_1}{1 + \cosh y_2} \right),$$

$$\frac{\partial}{\partial y_2} = \left(\frac{\cos y_1}{1 + \cosh y_2}, \frac{\sin y_1}{1 + \cosh y_2} \right).$$

Using the metric on the Poincaré unit ball model of hyperbolic space, we see that

$$\begin{aligned}\left\langle \frac{\partial}{\partial y_1}, \frac{\partial}{\partial y_1} \right\rangle &= \frac{4}{\left(1-\left(\frac{\sinh y_2}{1+\cosh y_2}\right)^2\right)^2} \frac{\sinh^2 y_2}{(1+\cosh y_2)^2} \\ &= \frac{4}{\left(\frac{2}{1+\cosh y_2}\right)^2} \frac{\sinh^2 y_2}{(1+\cosh y_2)^2} \\ &= \sinh^2 y_2,\end{aligned}$$

and similarly

$$\left\langle \frac{\partial}{\partial y_1}, \frac{\partial}{\partial y_2} \right\rangle = 0,$$
$$\left\langle \frac{\partial}{\partial y_2}, \frac{\partial}{\partial y_2} \right\rangle = 1.$$

6.3. Let $\mathbb{H}^2$ be the upper half-plane model of the hyperbolic 2-space.

(a) Let $0 < a < b$ and $c : [a,b] \to \mathbb{H}^2$, $c(t) = ti$. Calculate the arc-length reparametrization $\gamma : [0, \ln(b/a)] \to \mathbb{H}^2$.

(b) Let $c : [0,\pi] \to \mathbb{H}^2$, given by

$$c(t) = \frac{ai\cos t + \sin t}{-ai\sin t + \cos t},$$

for some $a > 1$. Calculate $L(c)$.

Solution:

(a) We have $c'(t) = i$ for all $t \in [a,b]$. The function $l : [a,b] \to [0, L(c)]$ is given by

$$l(t) = \int_a^t \|c'(s)\|_{c(s)}\, ds = \ln\frac{t}{a}.$$

Thus, $l : [a,b] \to [0, \ln(b/a)]$ is bijective, strictly monotone increasing and differentiable. We calculate its inverse:

$$s = l(t) \Leftrightarrow s = \ln\frac{t}{a} \Leftrightarrow e^s = \frac{t}{a} \Leftrightarrow t = ae^s.$$

Therefore, $l^{-1}(s) = ae^s$ and the arc length parametrization of c is given by $\gamma = c \circ l^{-1} : [0, \ln(b/a)] \to \mathbb{H}^2$,

$$\gamma(s) = c(l^{-1}(s)) = c(ae^s) = ae^s i.$$

(b) We have

$$c(t) = \frac{(ai\cos t + \sin t)(ai\sin t + \cos t)}{\cos^2 t + a^2\sin^2 t} = \frac{\sin t\cos t(1-a^2) + ia}{\cos^2 t + a^2\sin^2 t},$$

so

$$\mathrm{Im}(c(t)) = \frac{a}{\cos^2 t + a^2 \sin^2 t}.$$

On the other hand, we have

$$c'(t) = \frac{(-ai\sin t + \cos t)^2 + (ai\cos t + \sin t)^2}{(-ai\sin t + \cos t)^2} = \frac{1-a^2}{(-ai\sin t + \cos t)^2}.$$

This implies that

$$|c'(t)| = \frac{a^2-1}{\cos^2 t + a^2\sin^2 t},$$

and

$$\|c'(t)\|_{c(t)} = \frac{a^2-1}{\cos^2 t + a^2\sin^2 t} \frac{\cos^2 t + a^2\sin^2 t}{a} = \frac{a^2-1}{a} = a - \frac{1}{a}.$$

Thus, we obtain

$$L(c) = \int_0^\pi \|c'(t)\|_{c(t)}\, dt = \pi\left(a - \frac{1}{a}\right).$$

6.4. We work in the upper half-plane model of the hyperbolic 2-space $\mathbb{H}^2$. We will show that for $z_1, z_2 \in \mathbb{H}^2$ the distance function is given by the formula

$$\sinh(\frac{1}{2}d(z_1, z_2)) = \frac{|z_1 - z_2|}{2\sqrt{\mathrm{Im}(z_1)\mathrm{Im}(z_2)}}.$$

(a) Let $z_1 = iy_1$ and $z_2 = iy_2$ for $y_1, y_2 \in \mathbb{R}$. Verify that the formula holds in this case (you may use the formula for the distance between two such points derived in class).

(b) Let $A \in SL_2(\mathbb{R})$ and let $f_A(z)$ be the isometry of $\mathbb{H}^2$ considered in Exercise 5.4. Show that both sides of the formula are invariant under f_A (you may use the hint about $\mathrm{Im}(f_A(z))$ given in Exercise 5.4).

(c) Finally, given two points $z_1, z_2 \in \mathbb{H}^2$, find an $A \in SL_2(\mathbb{R})$ such that both $f_A(z_1)$ and $f_A(z_2)$ lie on the imaginary axis.

(d) Using what you know about Möbius transformations of $\mathbb{C}$, explain how you would draw the shortest path connecting two points $z_1, z_2 \in \mathbb{H}^2$.

Solution:

(a) In class we showed using elementary methods that the distance between $z_1 = iy_1$ and $z_2 = iy_2$ is $d(z_1, z_2) = \ln(y_1/y_2)$ (where we assume without loss of generality that $y_2 > y_1$).
In this case, the LHS of the equation to be verified is

$$\begin{aligned}\sinh(\frac{1}{2}d(z_1, z_2)) &= \frac{e^{\frac{\log(y_1/y_2)}{2}} - e^{\frac{-\log(y_1/y_2)}{2}}}{2} \\ &= \frac{e^{\log(\sqrt{y_1/y_2})} - e^{\log(\sqrt{y_2/y_1})}}{2} \\ &= \frac{\sqrt{y_1/y_2} - \sqrt{y_2/y_1}}{2}.\end{aligned}$$

And the RHS is

$$\begin{aligned}\frac{|z_1 - z_2|}{2\sqrt{\mathrm{Im}(z_1)\mathrm{Im}(z_2)}} &= \frac{y_1 - y_2}{2\sqrt{y_1 y_2}} \\ &= \frac{\sqrt{y_1/y_2} - \sqrt{y_2/y_1}}{2},\end{aligned}$$

which coincides with LHS.

(b) The LHS is preserved since isometries preserve distances (you can see this from the definition of isometry given in terms of the Riemannian metric and the definition of distance given as an infimum of the values of certain integrals).
The RHS requires some calculation using that

$$\mathrm{Im}(f_A(z)) = \frac{\mathrm{Im}(z)}{|cz + d|^2}.$$

We compute the RHS:

$$\begin{aligned}\frac{|f_A(z_1) - f_A(z_2)|}{2\sqrt{\mathrm{Im}(f_A(z_1))\mathrm{Im}(f_A(z_2))}} &= \frac{|cz_1 + d||cz_2 + d|\left|\left(\frac{az_1+b}{cz_1+d}\right) - \left(\frac{az_2+b}{cz_2+d}\right)\right|}{2\sqrt{\mathrm{Im}(z_1)\mathrm{Im}(z_2)}} \\ &= \frac{|(az_1 + b)(cz_2 + d) - (az_2 + b)(cz_1 + d)|}{2\sqrt{\mathrm{Im}(z_1)\mathrm{Im}(z_2)}} \\ &= \frac{|z_1(ad - bc) - z_2(ad - bc)|}{2\sqrt{\mathrm{Im}(z_1)\mathrm{Im}(z_2)}} \\ &= \frac{|z_1 - z_2|}{2\sqrt{\mathrm{Im}(z_1)\mathrm{Im}(z_2)}}\end{aligned}$$

which was required.

(c) You can achieve this by some basic Möbius transformations. If x_1, x_2 are real numbers $x_1 \neq x_2$, then

$$z \mapsto \frac{az - ax_2}{z - x_1}$$

takes x_2 to 0 and x_1 to ∞, where we choose $a \in \mathbb{R}$ so that the condition $\det A = 1$ is satisfied.

Suppose now that z_1 and z_2 are in the upper half-plane and lie on the unique semicircle or half-line through x_1 and x_2 which meets the real axis at right angles. Then this transformation must take z_1 and z_2 to the upper imaginary axis, since Möbius transformations map circles and lines to circles and lines.

(d) Möbius transformations take circles and lines to circles and lines and also preserve angles. Since a, b, c, d are all real, the class of Möbius transformations that we deal with all preserve the real axis. Since we know that vertical half-lines satisfy the shortest-distance property (see Example 3.14 from the lectures), we know that the only other curves which have a chance to are semicircles meeting the real axis at right angles. But (easy exercise!) given any two points in the upper half-plane, there is a unique semicircle or half-line through both points that meets the real axis at right angles.