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Michaelmas 2019

Riemannian Geometry IV, Homework 6 (Week 6)

Due date for starred problems: **Friday, November 22**.

6.1. Let X and Y be two vector fields on $\mathbb{R}^3$ defined by

$$\begin{aligned} X(x,y,z) &= z\frac{\partial}{\partial x} - 2z\frac{\partial}{\partial y} + (2y-x)\frac{\partial}{\partial z}, \\ Y(x,y,z) &= y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}, \end{aligned}$$

and let S^2 sit inside $\mathbb{R}^3$ as the sphere of radius 1 centered at the origin.

(a) Compute the Lie bracket $[X,Y]$.

(b) Verify that the restrictions of the vector fields X and Y to S^2 are vector fields on S^2 (in other words, are everywhere tangent to S^2).

(c) Check that the restriction of $[X,Y]$ to S^2 is also a vector field on S^2.

6.2. ($\star$) **Isometry between the hyperboloid and unit ball models of the hyperbolic plane**

Let $\mathbb{W}^2 = \{x \in \mathbb{R}^3 \mid q(x,x) = -1,\ x_3 > 0\}$ with $q(x,y) = x_1y_1 + x_2y_2 - x_3y_3$ be the hyperboloid model of the hyperbolic plane. Let the Poincaré unit ball model $\mathbb{B}^2$ of hyperbolic 2-space sit inside $\mathbb{R}^3$ as $\mathbb{B}^2 = \{x \in \mathbb{R}^3 \mid x_3 = 0,\ x_1^2 + x_2^2 < 1\}$.

We define a map $f : \mathbb{W}^2 \to \mathbb{B}^2$ by requiring that for each $p \in \mathbb{W}^2$ the points $f(p) \in \mathbb{B}^2$ and p are collinear with the point $(0,0,-1)$ (i.e. f is a projection from this point to the plane $\{z = 0\}$).

(a) Calculate explicitly the maps $f(X,Y,Z)$ for $(X,Y,Z) \in \mathbb{W}^2$ and $f^{-1}(x,y,0)$ for $(x,y,0) \in \mathbb{B}^2$.
Hint: you will (probably) obtain

$$x = \frac{X}{Z+1}, \quad y = \frac{Y}{Z+1}.$$

and

$$f^{-1}(x,y) = \left(\frac{2x}{1-x^2-y^2}, \frac{2y}{1-x^2-y^2}, \frac{1+x^2+y^2}{1-x^2-y^2}\right).$$

(b) An almost global coordinate chart $\varphi : U \to V$ on $\mathbb{W}^2$ is given by

$$\varphi^{-1}(x_1,x_2) = (\cos(x_1)\sinh(x_2), \sin(x_1)\sinh(x_2), \cosh(x_2)),$$

where $0 < x_1 < 2\pi$ and $0 < x_2 < \infty$. Let $\psi = \varphi \circ f^{-1}$ be a coordinate chart on $\mathbb{B}^2$ with coordinate functions y_1, y_2. Calculate ψ^{-1} explicitly.

(c) Explain why

$$Df(p)(\frac{\partial}{\partial x_i}) = \frac{\partial}{\partial y_i}$$

for for all $p \in U$ and $i = 1,2$, where $\frac{\partial}{\partial x_i} \in T_p\mathbb{W}^2$ and $\frac{\partial}{\partial y_i} \in T_{f(p)}\mathbb{B}^2$.

(d) Show that

$$\langle \frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j} \rangle_p = \langle \frac{\partial}{\partial y_i}, \frac{\partial}{\partial y_j} \rangle_{f(p)}$$

for all $p \in U$, and $i,j \in \{1,2\}$. Together with part (c), this demonstrates that f is an isometry.

Additional remark. To be precise, we need to choose two coordinate charts of the above type with $V_1 = (0, 2\pi) \times (0, \infty)$ and $V_2 = (-\pi, \pi) \times (0, \infty)$, and to consider also the linear map $Df(0, 0, 1) : T_{(0,0,1)}\mathbb{W}^2 \to T_0\mathbb{B}^2$ to cover the whole hyperbolic plane and to fully prove that f is an isometry.

6.3. Let $\mathbb{H}^2$ be the upper half-plane model of the hyperbolic 2-space.

(a) Let $0 < a < b$ and $c : [a, b] \to \mathbb{H}^2$, $c(t) = ti$. Calculate the arc-length reparametrization $\gamma : [0, \ln(b/a)] \to \mathbb{H}^2$.

(b) Let $c : [0, \pi] \to \mathbb{H}^2$, given by

$$c(t) = \frac{ai \cos t + \sin t}{-ai \sin t + \cos t},$$

for some $a > 1$. Calculate $L(c)$.

6.4. We work in the upper half-plane model of the hyperbolic 2-space $\mathbb{H}^2$. We will show that for $z_1, z_2 \in \mathbb{H}^2$ the distance function is given by the formula

$$\sinh(\frac{1}{2}d(z_1, z_2)) = \frac{|z_1 - z_2|}{2\sqrt{\mathrm{Im}(z_1)\mathrm{Im}(z_2)}}.$$

(a) Let $z_1 = iy_1$ and $z_2 = iy_2$ for $y_1, y_2 \in \mathbb{R}$. Verify that the formula holds in this case (you may use the formula for the distance between two such points derived in class).

(b) Let $A \in SL_2(\mathbb{R})$ and let $f_A(z)$ be the isometry of $\mathbb{H}^2$ considered in Exercise 5.4. Show that both sides of the formula are invariant under f_A (you may use the hint about $\mathrm{Im}(f_A(z))$ given in Exercise 5.4).

(c) Finally, given two points $z_1, z_2 \in \mathbb{H}^2$, find an $A \in SL_2(\mathbb{R})$ such that both $f_A(z_1)$ and $f_A(z_2)$ lie on the imaginary axis.

(d) Using what you know about Möbius transformations of $\mathbb{C}$, explain how you would draw the shortest path connecting two points $z_1, z_2 \in \mathbb{H}^2$.