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Riemannian Geometry IV, Solutions 7 (Week 7)

7.1. Covariant derivative in $\mathbb{R}^n$

We define covariant derivative $\nabla_v X$ of a vector field X in the direction of vector $v \in T_p\mathbb{R}^n = \mathbb{R}^n$ at point p in $\mathbb{R}^n$ as

$$(\nabla_v X)(p) = \lim_{t\to 0} \frac{X(p+tv) - X(p)}{t}$$

Show the following properties of the covariant derivative in $\mathbb{R}^n$:

(a) $\nabla_v(X+Y) = \nabla_v(X) + \nabla_v(Y)$;

(b) $\nabla_v(fX) = v(f)X(p) + f(p)\nabla_v X$, where $f \in \mathbb{C}^\infty(\mathbb{R}^n)$, and $v(f)$ denotes the derivative of f in direction v;

(c) $\nabla_{\alpha v+\beta w} X = \alpha\nabla_v X + \beta\nabla_w X$ for $\alpha, \beta \in \mathbb{R}$;

(d) $v(\langle X, Y\rangle) = \langle \nabla_v X, Y\rangle + \langle X, \nabla_v Y\rangle$, where $\langle\cdot,\cdot\rangle$ denotes the Euclidean dot-product, and $\langle X, Y\rangle$ is considered as a smooth function on $\mathbb{R}^n$;

(e) $\nabla_X Y - \nabla_y X = [X, Y]$, where $X, Y, \nabla_X Y, \nabla_Y X \in \mathfrak{X}(\mathbb{R}^n)$, and $(\nabla_X Y)(p)$ is defined as $(\nabla_{X(p)} Y)(p)$.

Solution:

We will check the equalities by computing in coordinates and using that

$$\nabla_v X = \sum_{i=1}^n v(a_i) \frac{\partial}{\partial x_i}\Big|_p$$

for $X = \sum_{i=1}^n a_i(p)\frac{\partial}{\partial x_i}$. Denote also $Y = \sum_{i=1}^n b_i(p)\frac{\partial}{\partial x_i}$.

(a) use that $v(a_i + b_i)(p) = v(a_i)(p) + v(b_i)(p)$;

(b) it is the Leibniz rule applied to each term of $\sum_{i=1}^n v(f\cdot a_i)\frac{\partial}{\partial x_i}$;

(c) derivatives form a vector space;

(d)

$$v(\langle X, Y\rangle) = v\big(\langle \sum_{i=1}^n a_i \frac{\partial}{\partial x_i}, \sum_{j=1}^n b_j \frac{\partial}{\partial x_j}\rangle\big) = v(\sum_{i=1}^n a_i b_i) =$$

$$\overset{\text{Leibniz rule}}{=} \sum_{i=1}^n (v(a_i)b_i(p) + a_i(p)v(b_i)) = \langle \nabla_v X, Y\rangle + \langle X, \nabla_v Y\rangle;$$

(e) $\nabla_X Y - \nabla_y X = \sum_{i=1}^n X(b_i)\frac{\partial}{\partial x_i} - \sum_{i=1}^n Y(a_i)\frac{\partial}{\partial x_i} = \sum_{i,j} a_j \frac{\partial b_i}{\partial x_j}\frac{\partial}{\partial x_i} - \sum_{i,j} b_j \frac{\partial a_i}{\partial x_j}\frac{\partial}{\partial x_i} = [X, Y]$

7.2. ($\star$) Let $\mathbb{H}^n$ be the upper half-space model of hyperbolic n-space,

$$\mathbb{H}^n = \{x \in \mathbb{R}^n \mid x_n > 0\}, \quad g(v, w) = \frac{\langle v, w \rangle}{x_n^2},$$

where $v, w \in T_x\mathbb{H}^n$, and we write g for the metric on $\mathbb{H}^n$ identifying each tangent space canonically with $\mathbb{R}^n$.

Calculate all Christoffel symbols Γ_{ij}^k for the global coordinate chart given by the identity map $\varphi : \mathbb{H}^n \to \mathbb{R}^n$, $\varphi(x) = x$.

Solution:

Using the standard global coordinate chart on $\mathbb{H}^n$, the matrix (g_{ij}) is diagonal with all diagonal entries $g_{ii} = 1/x_n^2$. We now use the formula

$$\Gamma_{ij}^k = \frac{1}{2} \sum_m g^{km}(g_{im,j} + g_{jm,i} - g_{ij,m}),$$

where

$$g_{ab,c} = \frac{\partial}{\partial x_c} g_{ab} \qquad \text{and} \qquad (g^{ij}) = (g_{ij})^{-1}$$

Since in our case the matrix (g_{ij}) is diagonal, its inverse is also diagonal, and thus we have

$$g^{ii} = x_n^2$$

Looking at the formula for Christoffel symbols, we see that we must have $k = m$ for any non-zero terms, and also at least one of i, j, k has to be equal to n.

More precisely, there are four cases giving potentially non-zero answers: $i = j \neq n, k = n$; $i = k \neq n, j = n$; $j = k \neq n, i = n$; $i = j = k = n$. Of these, the second and the third are really the same since $\Gamma_{ij}^k = \Gamma_{ji}^k$.

Then it is a simple matter of differentiation and we see that

$$\Gamma_{nn}^n = \frac{-1}{x_n} = \Gamma_{in}^i = \Gamma_{ni}^i, \quad \Gamma_{ii}^n = \frac{1}{x_n}$$

whenever $i \neq n$, with all other Christoffel symbols being 0.

7.3. (a) Calculate all Christoffel symbols Γ_{ij}^k for the unit ball model $\mathbb{B}^2$ of hyperbolic plane, again for the global coordinate chart given by the identity map $\varphi : \mathbb{B}^2 \to \mathbb{R}^2$, $\varphi(x) = x$. Recall the the metric is given by

$$g(v, w) = \frac{4}{(1 - \|x\|^2)^2} \langle v, w \rangle$$

(b) Do the same for the unit ball model $\mathbb{B}^n$ of hyperbolic n-space.

Solution:

(a) Parametrize the unit disc by

$$\varphi^{-1}(r, \vartheta) = (r \cos \vartheta, r \sin \vartheta), \qquad r \in (0, 1), \vartheta \in (0, 2\pi)$$

Then

$$\frac{\partial}{\partial r} = (\cos\vartheta, \sin\vartheta),$$
$$\frac{\partial}{\partial\vartheta} = (-r\sin\vartheta, r\cos\vartheta),$$

and thus we have

$$(g_{ij}) = \begin{pmatrix} \frac{4}{(1-r^2)^2} & 0 \\ 0 & \frac{4r^2}{(1-r^2)^2} \end{pmatrix},$$

which implies

$$g_{11,1} = \frac{16r}{(1-r^2)^3}, \quad g_{22,1} = \frac{8r(1+r^2)}{(1-r^2)^3},$$

and all the others $g_{ij,k}$ equal zero.
Now we can easily compute Christoffel symbols:

$$\Gamma^2_{11} = \Gamma^1_{12} = \Gamma^1_{21} = \Gamma^2_{22} = 0,$$

and

$$\Gamma^1_{11} = \frac{2r}{1-r^2}, \quad \Gamma^2_{12} = \Gamma^2_{21} = \frac{1+r^2}{r(1-r^2)}, \quad \Gamma^1_{22} = -\frac{r(1+r^2)}{1-r^2}$$

(b) The computations are similar to ones from previous exercises, but a bit bulky. One can parametrize a unit ball by

$$\varphi^{-1}(r, \vartheta_2, \ldots, \vartheta_n) = (r\sin\vartheta_n \ldots \sin\vartheta_2, r\sin\vartheta_n \ldots \sin\vartheta_3\cos\vartheta_2, r\sin\vartheta_n \ldots \sin\vartheta_4\cos\vartheta_3, \ldots, r\cos\vartheta_n),$$

where $r \in (0,1), \vartheta_2 \in (0, 2\pi), \vartheta_i \in (0,\pi)$ for $i > 2$. Then one can compute the metric (it will be diagonal, and g_{jj} depend on r and ϑ_i for $i > j$ only), and then compute the Christoffel symbols.