

Riemannian Geometry IV, Homework 7 (Week 7)

Due date for starred problems: **Friday, December 6.**

7.1. Covariant derivative in \mathbb{R}^n

We define covariant derivative $\nabla_v X$ of a vector field X in the direction of vector $v \in T_p \mathbb{R}^n = \mathbb{R}^n$ at point p in \mathbb{R}^n as

$$(\nabla_v X)(p) = \lim_{t \rightarrow 0} \frac{X(p + tv) - X(p)}{t}$$

Show the following properties of the covariant derivative in \mathbb{R}^n :

- (a) $\nabla_v(X + Y) = \nabla_v(X) + \nabla_v(Y)$;
- (b) $\nabla_v(fX) = v(f)X(p) + f(p)\nabla_v X$, where $f \in C^\infty(\mathbb{R}^n)$, and $v(f)$ denotes the derivative of f in direction v ;
- (c) $\nabla_{\alpha v + \beta w} X = \alpha \nabla_v X + \beta \nabla_w X$ for $\alpha, \beta \in \mathbb{R}$;
- (d) $v(\langle X, Y \rangle) = \langle \nabla_v X, Y \rangle + \langle X, \nabla_v Y \rangle$, where $\langle \cdot, \cdot \rangle$ denotes the Euclidean dot-product, and $\langle X, Y \rangle$ is considered as a smooth function on \mathbb{R}^n ;
- (e) $\nabla_X Y - \nabla_Y X = [X, Y]$, where $X, Y, \nabla_X Y, \nabla_Y X \in \mathfrak{X}(\mathbb{R}^n)$, and $(\nabla_X Y)(p)$ is defined as $(\nabla_{X(p)} Y)(p)$.

7.2. (★) Let \mathbb{H}^n be the upper half-space model of hyperbolic n -space,

$$\mathbb{H}^n = \{x \in \mathbb{R}^n \mid x_n > 0\}, \quad g(v, w) = \frac{\langle v, w \rangle_{\text{Eucl}}}{x_n^2},$$

where $v, w \in T_x \mathbb{H}^n$, and we write g for the metric on \mathbb{H}^n identifying each tangent space canonically with \mathbb{R}^n .

Calculate all Christoffel symbols Γ_{ij}^k for the global coordinate chart given by the identity map $\varphi : \mathbb{H}^n \rightarrow \mathbb{R}^n$, $\varphi(x) = x$.

7.3. (a) Calculate all Christoffel symbols Γ_{ij}^k for the unit ball model \mathbb{B}^2 of hyperbolic plane, again for the global coordinate chart given by the identity map $\varphi : \mathbb{B}^2 \rightarrow \mathbb{R}^2$, $\varphi(x) = x$. Recall the the metric is given by

$$g(v, w) = \frac{4}{(1 - \|x\|^2)^2} \langle v, w \rangle_{\text{Eucl}}$$

- (b) Do the same for the unit ball model \mathbb{B}^n of hyperbolic n -space.