Durham University
Pavel Tumarkin

Michaelmas 2019

Riemannian Geometry IV, Solutions 8 (Week 8)

8.1. Let $S^2 = \{(x, y, z) \in \mathbb{R}^3 \ : \ x^2 + y^2 + z^2 = 1\}$ be the unit sphere inside 3-space, with the induced metric from the standard Euclidean metric on $\mathbb{R}^3$.

(a) $(\star)$ Let c be the curve on S^2 given by

$$c(t) = \left(\frac{1}{\sqrt{2}}, \frac{\sin t}{\sqrt{2}}, \frac{\cos t}{\sqrt{2}}\right),$$

and let $v \in T_{c(0)}S^2$ be given by

$$v = (0, 1, 0) \in T_{c(0)}S^2 \subset T_{c(0)}\mathbb{R}^3.$$

Find the unique $X \in \mathfrak{X}_c(S^2)$ that is parallel along c and $X(0) = v$.

(b) Let $\gamma_1, \gamma_2 : [0, \pi] \to S^2$ be two curves connecting the north and south poles N and S defined by

$$\begin{aligned} \gamma_1(t) &= (0, \sin t, \cos t) \\ \gamma_2(t) &= (\sin t, 0, \cos t) \end{aligned}$$

Show that the isomorphisms of $T_N(S^2)$ and $T_S(S^2)$ given by parallel transports along γ_1 and γ_2 are different, i.e. find $u \in T_N(S^2)$ such that $P_{\gamma_1}(u) \neq P_{\gamma_2}(u)$.

Solution:

(a) We will compute using the following plan:

- write $X(t) = \sum a_i(t)\frac{\partial}{\partial x_i}$;
- calculate Christoffel symbols;
- use Γ^k_{ij} to find the action of the covariant derivative $\frac{D}{dt}$ on X;
- write a system of ODEs using the "parallel condition";
- solve it;
- find X.

In class we already computed the Christoffel symbols for S^2. Recall that we gave an almost global coordinate chart

$$\psi^{-1} : (\varphi, \vartheta) \mapsto (\cos\varphi \sin\vartheta, \sin\varphi \sin\vartheta_1, \cos\vartheta),$$

where $(\varphi, \vartheta) \in (0, 2\pi) \times (0, \pi)$. We calculated that

$$\Gamma^2_{11} = -\cos(\vartheta)\sin(\vartheta), \quad \Gamma^1_{12} = \Gamma^1_{21} = \cot(\vartheta)$$

with all other Christoffel symbols equal to 0.

Now, let us consider a similar chart:

$$\psi^{-1} : (\varphi, \vartheta) \mapsto (\cos\vartheta, \sin\varphi \sin\vartheta_1, \cos\varphi \sin\vartheta),$$

i.e. we interchange coordinates x and z. Clearly, this does not affect Christoffel symbols, but gives a better equation for the curve $c(t)$: we can write

$$c(t) = \psi^{-1}(t, \pi/4),$$

so that

$$c'(t) = \frac{\partial}{\partial\varphi}$$

Now we want to translate the "parallel condition" into a system of ODEs. So let $X(t) \in T_{c(t)}S^2$ be a vector field along the curve c. We can write

$$X(t) = a(t)\frac{\partial}{\partial\varphi} + b(t)\frac{\partial}{\partial\vartheta}$$

for some smooth functions a and b.
The parallel condition says that

$$\frac{D}{dt}X(t) = \frac{D}{dt}\left(a(t)\frac{\partial}{\partial\varphi} + b(t)\frac{\partial}{\partial\vartheta}\right) = 0,$$

and using the properties of $\frac{D}{dt}$ this is the same as requiring

$$a(t)\left(\nabla_{\frac{\partial}{\partial\varphi}}\frac{\partial}{\partial\varphi}\right) + a'(t)\frac{\partial}{\partial\varphi} + b(t)\left(\nabla_{\frac{\partial}{\partial\varphi}}\frac{\partial}{\partial\vartheta}\right) + b'(t)\frac{\partial}{\partial\vartheta} = 0$$

Here we need the Christoffel symbols. They tell us that

$$\begin{aligned}\nabla_{\frac{\partial}{\partial\varphi}}\frac{\partial}{\partial\vartheta} &= \cot(\vartheta)\frac{\partial}{\partial\varphi},\\ \nabla_{\frac{\partial}{\partial\varphi}}\frac{\partial}{\partial\varphi} &= -\cos(\vartheta)\sin(\vartheta)\frac{\partial}{\partial\vartheta}.\end{aligned}$$

Furthermore, since $\vartheta = \pi/4$ is constant on the curve c, our parallel condition becomes

$$-\frac{1}{2}a(t)\frac{\partial}{\partial\vartheta} + a'(t)\frac{\partial}{\partial\varphi} + b(t)\frac{\partial}{\partial\varphi} + b'(t)\frac{\partial}{\partial\vartheta} = 0$$

Since $\{\frac{\partial}{\partial\varphi}, \frac{\partial}{\partial\vartheta}\}$ form a basis of the tangent space at each point along c, we have

$$b'(t) - \frac{1}{2}a(t) = 0, \quad a'(t) + b(t) = 0$$

Solving this (and you definitely know how to do it), we get:

$$a(t) = A\cos(t/\sqrt{2}) + B\sin(t/\sqrt{2}), \quad b(t) = A\sqrt{2}\sin(t/\sqrt{2}) - B\sqrt{2}\cos(t\sqrt{2})$$

for arbitrary constants A and B. In our case we are told what $X(0)$ is, and that provides an initial condition so that we can find A and B. We have

$$X(0) = v = \sqrt{2}c'(0) = \sqrt{2}\frac{\partial}{\partial\varphi},$$

so we see that $A = \sqrt{2}$ and $B = 0$.
Hence,

$$X(t) = \sqrt{2}\cos(t/\sqrt{2})\frac{\partial}{\partial\varphi} + \sin(t/\sqrt{2})\frac{\partial}{\partial\vartheta}$$

This would be a good place to stop, but we can also write our field in three coordinates $(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z})$, so we observe that in terms of these ambient coordinates

$$\begin{aligned} \left.\frac{\partial}{\partial\varphi}\right|_{c(t)} &= \left(0, \frac{1}{\sqrt{2}}\cos t, -\frac{1}{\sqrt{2}}\sin t\right), \\ \left.\frac{\partial}{\partial\vartheta}\right|_{c(t)} &= \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\sin t, \frac{1}{\sqrt{2}}\cos t\right), \end{aligned}$$

and we can just substitute these into the expression that we already have:

$$X(t) = \sqrt{2}\cos(t/\sqrt{2})\left(0, \frac{1}{\sqrt{2}}\cos t, -\frac{1}{\sqrt{2}}\sin t\right) + \sin(t/\sqrt{2})\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\sin t, \frac{1}{\sqrt{2}}\cos t\right)$$

(b) Consider two vectors $v_1, v_2 \in T_N(S^2)$, $v_1 = (1,0,0) = \gamma_1'(0)$, $v_2 = (0,1,0) = \gamma_2'(0)$. We know that γ_1 is geodesic, so the field γ_1' is parallel along γ_1. In particular, $P_{\gamma_1}(v_1) = \gamma_1'(\pi) = (-1,0,0)$.
Note that by Prop. 4.18 from the lectures P_γ is a linear isometry for any curve γ (see also Exercise 8.3). In particular, if $X(t)$ is a parallel vector field along γ_1 with $X(0) = v_2 = (0,1,0)$, the vectors $\gamma_1'(t)$ and $X(t)$ form an orthonormal basis of $T_{\gamma_1(t)}S^2$. By continuity, one can see that $X(t) \equiv (0,1,0)$, and, in particular, $P_{\gamma_1}(v_2) = (0,1,0)$. Now, since γ_2 is geodesic, the field γ_2' is parallel along γ_2, so $P_{\gamma_2}(v_2) = \gamma_2'(\pi) = (0,-1,0) \neq P_{\gamma_1}(v_2)$.

8.2. Let $\mathbb{H}^2 = \{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}$ be the upper-half plane with its usual hyperbolic metric. Let c be the curve in $\mathbb{H}^2$ given by $c(t) = i + t$ for $t \in \mathbb{R}$. Identifying the tangent space to each point of $\mathbb{H}^2$ in the usual way with $\mathbb{C}$, find the parallel vector field $X(t) \in \mathbb{C} = T_{c(t)}\mathbb{H}^2$ along c, which is determined by its value at $t = 0$:

$$X(0) = 1 \in \mathbb{C} = T_i\mathbb{H}^2.$$

Solution: This question follows the same lines as the Exercise 8.1(a), so we move a bit faster.

Let

$$X(t) = a(t)\frac{\partial}{\partial x} + b(t)\frac{\partial}{\partial y}$$

be a parallel vector field along the curve c. Now $c'(t) = \frac{\partial}{\partial x}$, so the parallel condition becomes

$$a'(t)\frac{\partial}{\partial x} + a(t)\left(\nabla_{\frac{\partial}{\partial x}}\frac{\partial}{\partial x}\right) + b'(t)\frac{\partial}{\partial y} + b(t)\left(\nabla_{\frac{\partial}{\partial x}}\frac{\partial}{\partial y}\right) = 0$$

In Exercise 7.2 we computed the Christoffel symbols for the hyperbolic plane, so we know that

$$\nabla_{\frac{\partial}{\partial x}}\frac{\partial}{\partial x} = \frac{1}{y}\frac{\partial}{\partial y} \quad \text{and} \quad \nabla_{\frac{\partial}{\partial x}}\frac{\partial}{\partial y} = \frac{-1}{y}\frac{\partial}{\partial x}$$

Furthermore, the y-coordinate is fixed along c by $y = 1$. Thus, the parallel condition is equivalent to the following system of ODEs:

$$a'(t) - b(t) = 0, \ \ b'(t) + a(t) = 0,$$

which has solution

$$a(t) = A\cos t + B\sin t, \quad b(t) = -A\sin t + B\cos t$$

for arbitrary constants A, B. Now we know that $X(0) = \frac{\partial}{\partial x}$, so we can find the constants $A = 1, B = 0$. Thus,

$$X(t) = \cos t\frac{\partial}{\partial x} - \sin t\frac{\partial}{\partial y}$$

8.3. Let (M, g) be a Riemannian manifold and $c : [a, b] \to M$ be a smooth curve. Let $\frac{D}{dt}$ denote the corresponding covariant derivative along the curve c. Let X, Y be any two parallel vector fields X, Y along c. Show that

$$\frac{d}{dt}\langle X, Y\rangle \equiv 0,$$

i.e., the parallel transport $P_c : T_{c(a)}M \to T_{c(b)}M$ is a linear isometry.

(a) ($\star$) Prove this statement in the particular case when the vector fields X, Y along c have global extensions $\tilde{X}, \tilde{Y} : M \to TM$.

(b) Do the same computation for a general case writing $X(t)$, $Y(t)$ in local coordinates.

Solution:

(a) Assume first that there are global vector fields $\tilde{X}, \tilde{Y} : M \to TM$ with $\tilde{X}(c(t)) = X(t)$ and $\tilde{Y}(c(t)) = Y(t)$ for all $t \in [a, b]$. Since the Levi-Civita connection is Riemannian, we conclude that

$$\frac{d}{dt}\Big|_t \langle X, Y\rangle = \frac{d}{dt}\Big|_t \left(\langle \tilde{X}, \tilde{Y}\rangle \circ c\right) = c'(t)\left(\langle \tilde{X}, \tilde{Y}\rangle\right) =$$
$$= \langle \nabla_{c'(t)}\tilde{X}, Y(t)\rangle + \langle X(t), \nabla_{c'(t)}\tilde{Y}\rangle = \langle \frac{D}{dt}X(t), Y(t)\rangle + \langle X(t), \frac{D}{dt}Y(t)\rangle = 0$$

since the vector fields X, Y are parallel along c. But this implies that $t \mapsto \langle X(t), Y(t)\rangle$ is a constant function, i.e. the parallel transport $P_c : T_{c(a)}M \to T_{c(b)}M$ is an isometry, since

$$\langle P_c X(a), P_c Y(a)\rangle = \langle X(b), Y(b)\rangle = \langle X(a), Y(a)\rangle.$$

(b) Now we assume that X, Y do not have global extensions. Assume that there is a coordinate chart $\varphi : (x_1, \ldots, x_n) : U \to V$ with $c([a, b]) \subset U$. Then we can write

$$X(t) = \sum a_j(t)\frac{\partial}{\partial x_j}\Big|_{c(t)}, \quad Y(t) = \sum b_j(t)\frac{\partial}{\partial x_j}\Big|_{c(t)},$$

and we have

$$\frac{d}{dt}\langle X, Y\rangle = \frac{d}{dt}\left(\sum_{j,k} a_j b_k \left(\langle \frac{\partial}{\partial x_j}, \frac{\partial}{\partial x_k}\rangle \circ c\right)\right).$$

As before, the Riemannian property of the Levi-Civita connection yields

$$\frac{d}{dt}\left(\langle \frac{\partial}{\partial x_j}, \frac{\partial}{\partial x_k}\rangle \circ c\right) = \langle \nabla_{c'(t)}\frac{\partial}{\partial x_j}, \frac{\partial}{\partial x_k}\Big|_{c(t)}\rangle + \langle \frac{\partial}{\partial x_j}\Big|_{c(t)}, \nabla_{c'(t)}\frac{\partial}{\partial x_k}\rangle.$$

This implies that

$$\frac{d}{dt}\langle X, Y\rangle = \sum_{j,k}(a_j' b_k + a_j b_k')\langle \frac{\partial}{\partial x_j}, \frac{\partial}{\partial x_k}\rangle \circ c + a_j b_k \left(\langle \nabla_{c'(t)}\frac{\partial}{\partial x_j}, \frac{\partial}{\partial x_k}\Big|_{c(t)}\rangle + \langle \frac{\partial}{\partial x_j}\Big|_{c(t)}, \nabla_{c'(t)}\frac{\partial}{\partial x_k}\rangle\right)$$
$$= \langle \sum_j a_j'(t)\frac{\partial}{\partial x_j}\Big|_{c(t)} + a_j(t)\nabla_{c'(t)}\frac{\partial}{\partial x_j}, \sum_k b_k(t)\frac{\partial}{\partial x_k}\Big|_{c(t)}\rangle +$$
$$+ \langle \sum_j a_j(t)\frac{\partial}{\partial x_j}\Big|_{c(t)}, \sum_k b_k'(t)\frac{\partial}{\partial x_k}\Big|_{c(t)} + b_k(t)\nabla_{c'(t)}\frac{\partial}{\partial x_k}\rangle =$$
$$= \langle \sum_j \frac{D}{dt}\left(a_j\frac{\partial}{\partial x_j} \circ c\right), Y\rangle + \langle X, \sum_k \frac{D}{dt}\left(a_k\frac{\partial}{\partial x_k} \circ c\right) = \langle \frac{D}{dt}X, Y\rangle + \langle X, \frac{D}{dt}Y\rangle = 0.$$

Finally, if we need k coordinate charts $U_1, \dots, U_k$ to cover $c([a,b])$, i.e., if we have

$$c([a,b]) \subset \bigcup_{j=1}^{k} U_j$$

with a partition $a < t_1 < t_2 \cdots < t_{k-1} < b$ such that $c(a), c(t_1) \in U_1, c(t_1), c(t_2) \in U_2, \dots, c(t_{k-1}), c(b) \in U_k$, we conclude with the previous argument that $\frac{d}{dt}\langle X, Y\rangle$ is constant on the segments $[a, t_1], [t_1, t_2], \dots, [t_{k-1}, b]$, and therefore, constant on all $[a,b]$.

8.4. Given a curve $c : [a,b] \to \mathbb{R}^3$, $c(t) = (f(t), 0, g(t))$ without self-intersections and with $f(t) > 0$ for all $t \in [a,b]$, let $M \subset \mathbb{R}^3$ denote the surface of revolution obtained by rotating this curve around the z-axis. Let ∇ denote the Levi-Civita connection of M. An almost global coordinate chart is given by $\varphi : U \to V := (a,b) \times (0, 2\pi)$,

$$\varphi^{-1}(x_1, x_2) = (f(x_1)\cos x_2, f(x_1)\sin x_2, g(x_1)).$$

(a) Calculate the Christoffel symbols of this coordinate chart and express $\nabla_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_j}$ in terms of the basis $\frac{\partial}{\partial x_k}$.

(b) Let $\gamma_1(t) = \varphi^{-1}(x_1 + t, x_2)$. Calculate

$$\frac{D}{dt}\gamma_1',$$

where $\frac{D}{dt}$ denotes the covariant derivative along γ_1. Show that this vector field along γ_1 vanishes if and only if the generating curve c of M is parametrized proportionally to arc-length. Note that γ_1 is obtained by rotation of c by a fixed angle. Derive from these facts that meridians of a surface of revolution are geodesics if they are parametrized proportionally to arc length.

(c) Let $\gamma_2(t) = \varphi^{-1}(x_1, x_2 + t)$. Calculate

$$\frac{D}{dt}\gamma_2',$$

where $\frac{D}{dt}$ denotes the covariant derivative along γ_2. Show that this vector field along γ_2 vanishes if and only if $f'(x_1) = 0$. Explain that this implies that parallels of a surface of revolution are geodesics if they have locally maximal or minimal radius.

Solution:

(a) We have

$$\begin{aligned}
\frac{\partial}{\partial x_1}\Big|_{\varphi^{-1}(x_1,x_2)} &= (f'(x_1)\cos x_2, f'(x_1)\sin x_2, g'(x_1)), \\
\frac{\partial}{\partial x_2}\Big|_{\varphi^{-1}(x_1,x_2)} &= (-f(x_1)\sin x_2, f(x_1)\cos x_2, 0).
\end{aligned}$$

This implies that

$$(g_{ij}) = \begin{pmatrix} (f'(x_1))^2 + (g'(x_1))^2 & 0 \\ 0 & f^2(x_1) \end{pmatrix} = \begin{pmatrix} \|c'(x_1)\|^2 & 0 \\ 0 & f^2(x_1) \end{pmatrix}$$

and

$$(g^{ij}) = \begin{pmatrix} \frac{1}{(f'(x_1))^2 + (g'(x_1))^2} & 0 \\ 0 & \frac{1}{f^2(x_1)} \end{pmatrix}.$$

Consequently, we have

$$\begin{aligned} g_{11,1} &= 2(f'(x_1)f''(x_1) + g'(x_1)g''(x_1)), \\ g_{22,1} &= 2f(x_1)f'(x_1), \end{aligned}$$

and the Christoffel symbols are calculated as

$$\begin{aligned} \Gamma_{11}^1 &= \frac{1}{2}g^{11}\left(g_{11,1} + g_{11,1} - g_{11,1}\right) = \frac{f'(x_1)f''(x_1) + g'(x_1)g''(x_1)}{(f'(x_1))^2 + (g'(x_1))^2}, \\ \Gamma_{11}^2 &= \frac{1}{2}g^{22}(g_{12,1} + g_{12,1} - g_{11,2}) = 0, \\ \Gamma_{12}^1 &= \frac{1}{2}g^{11}(g_{11,2} + g_{12,1} - g_{12,1}) = 0 = \Gamma_{21}^1, \\ \Gamma_{12}^2 &= \frac{1}{2}g^{22}(g_{12,2} + g_{22,1} - g_{12,2}) = \frac{f'(x_1)}{f(x_1)} = \Gamma_{21}^2, \\ \Gamma_{22}^1 &= \frac{1}{2}g^{11}(g_{21,2} + g_{21,2} - g_{22,1}) = \frac{-f(x_1)f'(x_1)}{(f'(x_1))^2 + (g'(x_1))^2}, \\ \Gamma_{22}^2 &= \frac{1}{2}g^{22}(g_{22,2} + g_{22,2} - g_{22,2}) = 0. \end{aligned}$$

This implies that

$$\begin{aligned} \nabla_{\frac{\partial}{\partial x_1}}\frac{\partial}{\partial x_1} &= \frac{f'(x_1)f''(x_1) + g'(x_1)g''(x_1)}{(f'(x_1))^2 + (g'(x_1))^2}\frac{\partial}{\partial x_1}, \\ \nabla_{\frac{\partial}{\partial x_1}}\frac{\partial}{\partial x_2} &= \frac{f'(x_1)}{f(x_1)}\frac{\partial}{\partial x_2}, \\ \nabla_{\frac{\partial}{\partial x_2}}\frac{\partial}{\partial x_1} &= \frac{f'(x_1)}{f(x_1)}\frac{\partial}{\partial x_2}, \\ \nabla_{\frac{\partial}{\partial x_2}}\frac{\partial}{\partial x_2} &= \frac{-f(x_1)f'(x_1)}{(f'(x_1))^2 + (g'(x_1))^2}\frac{\partial}{\partial x_1} \end{aligned}$$

(b) Note that we have

$$\gamma_1'(t) = \frac{\partial}{\partial x_1}|_{\gamma_1(t)}.$$

This implies that

$$\begin{aligned} \left(\frac{D}{dt}\gamma_1'\right)(t) = \nabla_{\gamma_1'(t)}\frac{\partial}{\partial x_1} = \left(\nabla_{\frac{\partial}{\partial x_1}}\frac{\partial}{\partial x_1}\right)(\gamma_1(t)) = \\ = \frac{f'(x_1+t)f''(x_1+t) + g'(x_1+t)g''(x_1+t)}{(f'(x_1+t))^2 + (g'(x_1+t))^2}\frac{\partial}{\partial x_1}|_{\gamma_1(t)} \in T_{\gamma_1(t)}M. \end{aligned}$$

The condition $\frac{D}{dt}\gamma_1' \equiv 0$ is equivalent to $f'(t)f''(t) + g'(t)g''(t) = 0$ for all $t \in (a,b)$, which in its turn is equivalent to $(f'(t))^2 + (g'(t))^2 = \text{const}$. Since

$$\|c'(t)\|^2 = (f'(t))^2 + (g'(t))^2,$$

we conclude that $\frac{D}{dt}\gamma_1'$ vanishes identically if and only if c is parametrized proportionally to arc length. Since c and γ_1 are obtained from each other by an isometry of $\mathbb{R}^3$, namely a rotation by the angle x_2 around the z-axis, c is parametrized proportionally to arc length if and only if γ_1 is parametrized proportionally to arc length.

(c) We have

$$\gamma_2'(t) = \frac{\partial}{\partial x_2}|_{\gamma_2(t)}.$$

This implies that

$$\left(\frac{D}{dt}\gamma_2'\right)(t) = \nabla_{\gamma_2'(t)}\frac{\partial}{\partial x_2} = \left(\nabla_{\frac{\partial}{\partial x_2}}\frac{\partial}{\partial x_2}\right)(\gamma_2(t)) = \frac{-f(x_1)f'(x_1)}{(f'(x_1))^2+(g'(x_1))^2}\frac{\partial}{\partial x_1}|_{\gamma_2(t)} \in T_{\gamma_2(t)}M.$$

Since $f > 0$, the condition $\frac{D}{dt}\gamma_2' \equiv 0$ is equivalent to $f'(x_1) = 0$, which holds, in particular, if f has a local maximum or minimum at x_1. Now observe that γ_2 is a parallel of the surface of revolution M, and $f(x_1)$ is its radius (i.e., the distance to the z-axis).