Durham University
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Riemannian Geometry IV, Homework 8 (Week 8)

Due date for starred problems: **Friday, December 6**.

8.1. Let $S^2 = \{(x, y, z) \in \mathbb{R}^3 \ : \ x^2 + y^2 + z^2 = 1\}$ be the unit sphere inside 3-space, with the induced metric from the standard Euclidean metric on $\mathbb{R}^3$.

(a) ($\star$) Let c be the curve on S^2 given by

$$c(t) = \left(\frac{1}{\sqrt{2}}, \frac{\sin t}{\sqrt{2}}, \frac{\cos t}{\sqrt{2}} \right),$$

and let $v \in T_{c(0)}S^2$ be given by

$$v = (0, 1, 0) \in T_{c(0)}S^2 \subset T_{c(0)}\mathbb{R}^3.$$

Find the unique $X \in \mathfrak{X}_c(S^2)$ that is parallel along c and $X(0) = v$.

(b) Let $\gamma_1, \gamma_2 : [0, \pi] \to S^2$ be two curves connecting the north and south poles N and S defined by

$$\begin{aligned} \gamma_1(t) &= (0, \sin t, \cos t) \\ \gamma_2(t) &= (\sin t, 0, \cos t) \end{aligned}$$

Show that the isomorphisms of $T_N(S^2)$ and $T_S(S^2)$ given by parallel transports along γ_1 and γ_2 are different, i.e. find $u \in T_N(S^2)$ such that $P_{\gamma_1}(u) \neq P_{\gamma_2}(u)$.

8.2. Let $\mathbb{H}^2 = \{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}$ be the upper-half plane with its usual hyperbolic metric. Let c be the curve in $\mathbb{H}^2$ given by $c(t) = i + t$ for $t \in \mathbb{R}$. Identifying the tangent space to each point of $\mathbb{H}^2$ in the usual way with $\mathbb{C}$, find the parallel vector field $X(t) \in \mathbb{C} = T_{c(t)}\mathbb{H}^2$ along c, which is determined by its value at $t = 0$:

$$X(0) = 1 \in \mathbb{C} = T_i\mathbb{H}^2.$$

8.3. Let (M, g) be a Riemannian manifold and $c : [a, b] \to M$ be a smooth curve. Let $\frac{D}{dt}$ denote the corresponding covariant derivative along the curve c. Let X, Y be any two parallel vector fields X, Y along c. Show that

$$\frac{d}{dt}\langle X, Y \rangle \equiv 0,$$

i.e., the parallel transport $P_c : T_{c(a)}M \to T_{c(b)}M$ is a linear isometry.

(a) ($\star$) Prove this statement in the particular case when the vector fields X, Y along c have global extensions $\tilde{X}, \tilde{Y} : M \to TM$.

(b) Do the same computation for a general case writing $X(t)$, $Y(t)$ in local coordinates.

8.4. Given a curve $c : [a, b] \to \mathbb{R}^3$, $c(t) = (f(t), 0, g(t))$ without self-intersections and with $f(t) > 0$ for all $t \in [a, b]$, let $M \subset \mathbb{R}^3$ denote the surface of revolution obtained by rotating this curve around the z-axis. Let ∇ denote the Levi-Civita connection of M. An almost global coordinate chart is given by $\varphi : U \to V := (a, b) \times (0, 2\pi)$,

$$\varphi^{-1}(x_1, x_2) = (f(x_1) \cos x_2, f(x_1) \sin x_2, g(x_1)).$$

(a) Calculate the Christoffel symbols of this coordinate chart and express $\nabla_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_j}$ in terms of the basis $\frac{\partial}{\partial x_k}$.

(b) Let $\gamma_1(t) = \varphi^{-1}(x_1 + t, x_2)$. Calculate

$$\frac{D}{dt}\gamma_1',$$

where $\frac{D}{dt}$ denotes the covariant derivative along γ_1. Show that this vector field along γ_1 vanishes if and only if the generating curve c of M is parametrized proportional to arc-length. Note that γ_1 is obtained by rotation of c by a fixed angle. Derive from these facts that meridians of a surface of revolution are geodesics if they are parametrized proportional to arc length.

(c) Let $\gamma_2(t) = \varphi^{-1}(x_1, x_2 + t)$. Calculate

$$\frac{D}{dt}\gamma_2',$$

where $\frac{D}{dt}$ denotes the covariant derivative along γ_2. Show that this vector field along γ_2 vanishes if and only if $f'(x_1) = 0$. Explain that this implies that parallels of a surface of revolution are geodesics if they have locally maximal or minimal radius.