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Riemannian Geometry IV, Solutions 9 (Week 9)

9.1. First Variation Formula of energy.
Let $F : (-\varepsilon, \varepsilon) \times [a, b] \to M$ be a variation of a smooth curve $c : [a, b] \to M$ with $c'(t) \neq 0$ for all $t \in [a, b]$, and let X be its variational vector field. Let $E : (-\varepsilon, \varepsilon) \to \mathbb{R}_+$ denote the associated energy, i.e.,

$$E(s) = \frac{1}{2}\int_a^b \|\frac{\partial F}{\partial t}(s,t)\|^2 dt.$$

(a) Show that

$$E'(0) = \langle X(b), c'(b)\rangle - \langle X(a), c'(a)\rangle - \int_a^b \langle X(t), \frac{D}{dt}c'(t)\rangle dt.$$

Simplify the formula for the cases when

(b) c is a geodesic,

(c) F is a proper variation,

(d) c is a geodesic and F is a proper variation.

Let $c : [a, b] \to M$ be a curve connecting p and q (not necessarily parametrized proportional to arc length). Show that

(e) $E'(0) = 0$ for every proper variation implies that c is a geodesic.

(f) Assume that c minimizes the energy amongst all curves $\gamma : [a, b] \to M$ which connect p and q. Then c is a geodesic.

Solution:

(a) We have

$$E'(0) = \frac{d}{ds}\Big|_{s=0} \frac{1}{2}\int_a^b \|\frac{\partial F}{\partial t}(s,t)\|^2 dt = \frac{1}{2}\int_a^b \frac{\partial}{\partial s}\Big|_{s=0} \langle \frac{\partial F}{\partial t}(s,t), \frac{\partial F}{\partial t}(s,t)\rangle dt = \int_a^b \langle \frac{D}{ds}\frac{\partial F}{\partial t}(0,t), c'(t)\rangle dt.$$

Applying the Symmetry Lemma yields

$$E'(0) = \int_a^b \langle \frac{D}{dt}\frac{\partial F}{\partial s}(0,t), c'(t)\rangle dt = \int_a^b \frac{d}{dt}\langle X(t), c'(t)\rangle - \langle X(t), \frac{D}{dt}c'(t)\rangle dt =$$
$$= \langle X(b), c'(b)\rangle - \langle X(a), c'(a)\rangle - \int_a^b \langle X(t), \frac{D}{dt}c'(t)\rangle dt.$$

(b) If c is a geodesic, this simplifies to $E'(0) = \langle X(b), c'(b)\rangle - \langle X(a), c'(a)\rangle$.

(c) If F is a proper variation, this simplifies to $E'(0) = -\int_a^b \langle X(t), \frac{D}{dt}c'(t)\rangle dt$.

(d) If c is a geodesic and F is a proper variation, this simplifies to $E'(0) = 0$.

(e) Assume that c is not a geodesic. Then there exists a $t_0 \in (a,b)$ with $\frac{D}{dt}c'(t_0) \neq 0$ (since the map $t \mapsto \frac{D}{dt}c'(t_0)$ is continuous). Choose a smooth function $\varphi : [a,b] \to \mathbb{R}_{\geq 0}$ with $\varphi(a) = \varphi(b) = 0$ and $\varphi(t_0) = 1$ and set $X(t) = \varphi(t)\frac{D}{dt}c'(t)$. Then X is the variational vector field of some proper variation F, and we obtain for its energy functional

$$E'(0) = -\int_a^b \langle X(t), \frac{D}{dt}c'(t)\rangle dt = -\int_a^b \varphi(t)\|\frac{D}{dt}c'(t)\|dt < 0.$$

Therefore,

$$c \text{ is not geodesic} \quad \Rightarrow \quad E'(0) \neq 0 \text{ for some proper variation.}$$

(f) Now assume that c minimizes energy amongst all curves $\gamma : [a,b] \to M$ connecting p and q. Let F be a proper variation. Then the curves $t \mapsto F(s,t)$ are also curves $[a,b] \to M$ connecting p, q, so their energy is $\geq E(0) = E(c)$. This implies that $E'(0) = 0$. Using (e), we conclude that c is a geodesic.

9.2. Rescaling Lemma.
Let $c : [0,a] \to M$ be a geodesic, and $k > 0$. Define a curve γ by

$$\gamma : [0, a/k] \to M, \qquad \gamma(t) = c(kt)$$

Show that γ is geodesic with $\gamma'(t) = kc'(kt)$.

Solution: Proof is sraightforward: all the entries of the corresponding differential equation for $c(t)$ are multiplied by k^2.

9.3. Let M be a smooth manifold, let $\mathfrak{X}(M)$ be the vector space of smooth vector fields on M, and ∇ be a *general* affine connection (we do not require a Riemannian metric on M and the "Riemannian property", neither the "torsion-free property" of the Levi-Civita connection). We say a map

$$A : \mathfrak{X}(M) \times \cdots \times \mathfrak{X}(M) \to C^\infty(M) \text{ or } \mathfrak{X}(M)$$

is a *tensor* if it is linear in each argument, i.e.,

$$A(X_1, \cdots, fX_i + gY_i, \cdots, X_r) = fA(X_1, \cdots, X_i, \cdots, X_r) + gA(X_1, \cdots, Y_i, \cdots, X_r),$$

for all $X, Y \in \mathfrak{X}(M)$ and $f, g \in C^\infty(M)$.

(a) Show that

$$T : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M), \qquad T(X,Y) = [X,Y] - (\nabla_X Y - \nabla_Y X)$$

is a tensor (called the *torsion* of the manifold M).

(b) Let

$$A : \underbrace{\mathfrak{X}(M) \times \cdots \times \mathfrak{X}(M)}_{r \text{ factors}} \to C^\infty(M)$$

be a tensor. The covariant derivative of A is a map

$$\nabla A : \underbrace{\mathfrak{X}(M) \times \cdots \times \mathfrak{X}(M)}_{r+1 \text{ factors}} \to C^\infty(M),$$

defined by

$$\nabla A(X_1, \ldots, X_r, Y) = Y(A(X_1, \ldots, X_r)) - \sum_{j=1}^{r} A(X_1, \ldots, \nabla_Y X_j, \ldots, X_r).$$

Show that ∇A is a tensor.

(c) Let (M, g) be a Riemannian manifold and $G : \mathfrak{X}(M) \times \mathfrak{X}(M) \to C^\infty(M)$ be the Riemannian tensor, i.e., $G(X, Y) = \langle X, Y\rangle$. Calculate ∇G. What does it mean that $\nabla G \equiv 0$?

Solution:

(a) Note that $T(X, Y) = -T(Y, X)$, so we only have to prove linearity in the first argument. Moreover, we obviously have $T(X_1 + X_2, Y) = T(X_1, Y) + T(X_2, Y)$. We are left to show that

$$T(fX, Y) = fT(X, Y).$$

The calculation for this goes as follows:

$$\begin{aligned} T(fX, Y) = [fX, Y] - (\nabla_{fX}Y - \nabla_Y fX) = f[X, Y] - (Yf)X - (f\nabla_X Y - (Yf)X - f\nabla_Y X) = \\ = f([X, Y] - (\nabla_X Y - \nabla_Y X)) - (Yf)X + (Yf)X = fT(X, Y). \end{aligned}$$

(b) It is, again, straightforward to check that

$$\nabla A(X_1, \ldots, X_i + \tilde{X}_i, \ldots, X_r, X_{r+1}) = \nabla A(X_1, \ldots, X_i, \ldots, X_r, X_{r+1}) + \nabla A(X_1, \ldots, \tilde{X}_i, \ldots, X_r, X_{r+1})$$

for $i = 1, 2, \ldots, r + 1$. So it remains to show that

$$\nabla A(X_1, \ldots, fX_i, \ldots, X_r, X_{r+1}) = f\nabla A(X_1, \ldots, X_i, \ldots, X_r, X_{r+1}),$$

for $i = 1, 2, \ldots, r + 1$. Let $i = 1, 2, \ldots, r$. Then

$$\begin{aligned} &\nabla A(X_1, \ldots, fX_i, \ldots, X_r, Y) = \\ &\quad = Y(fA(X_1, \ldots, X_r)) - f\sum_{j=1}^{n} A(X_1, \ldots, \nabla_Y X_j, \ldots, X_r) - (Yf)A(X_1, \ldots, X_r) = \\ &\quad = fY(A(X_1, \ldots, X_r)) - f\sum_{j=1}^{n} A(X_1, \ldots, \nabla_Y X_j, \ldots, X_r) = f\nabla A(X_1, \ldots, X_r, Y). \end{aligned}$$

Finally, we obtain

$$\begin{aligned} \nabla A(X_1, \ldots, X_r, fY) = fY(A(X_1, \ldots, X_2)) - \sum A(X_1, \ldots, f\nabla_Y X_j, \ldots, X_r) = \\ = f\nabla A(X_1, \ldots, X_r, Y). \end{aligned}$$

(c) Using (b), we obtain

$$\nabla G(X, Y, Z) = Z(\langle X, Y\rangle) - \langle\nabla_Z X, Y\rangle - \langle X, \nabla_Z Y\rangle.$$

Then $\nabla G \equiv 0$ means precisely that the affine connection ∇ has the "Riemannian property".