Durham University
Pavel Tumarkin

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Riemannian Geometry IV, Term 1 (Section 1)

1 Smooth manifolds

"Smooth" means "infinitely differentiable", C^∞.

Definition 1.1. Let M be a set. An n-dimensional smooth atlas on M is a collection of triples $(U_\alpha, V_\alpha, \varphi_\alpha)$, where $\alpha \in I$ for some indexing set I, s.t.

(a) $U_\alpha \subseteq M$; $V_\alpha \subseteq \mathbb{R}^n$ is open $\forall \alpha \in I$;

(b) $\bigcup_{\alpha \in I} U_\alpha = M$;

(c) Each $\varphi_\alpha : U_\alpha \to V_\alpha$ is a bijection;

(d) For every $\alpha, \beta \in I$ such that $U_\alpha \cap U_\beta \neq \emptyset$ the composition $\varphi_\beta \circ \varphi_\alpha^{-1}\big|_{\varphi_\alpha(U_\alpha \cap U_\beta)} : \varphi_\alpha(U_\alpha \cap U_\beta) \to \varphi_\beta(U_\alpha \cap U_\beta)$ is a smooth map for all ordered pairs (α, β), where $\alpha, \beta \in I$.

The number n is called the dimension of M, the maps φ_α are called coordinate charts, the compositions $\varphi_\beta \circ \varphi_\alpha^{-1}$ are called transition maps or changes of coordinates.

Example 1.2. Two atlases on a circle $S^1 \subset \mathbb{R}^2$.

Definition 1.3. Let M have a smooth atlas. A set $A \subseteq M$ is open if for every $\alpha \in I$ the set $\varphi_\alpha(A \cap U_\alpha)$ is open in $\mathbb{R}^n$. If $A \subset M$ is open and $x \in A$, A is called an open neighborhood of x.

Definition 1.4. M is called Hausdorff if for each $x, y \in M$, $x \neq y$, there exist open sets $A_x \ni x$ and $A_y \ni y$ such that $A_x \cap A_y = \emptyset$.

Example 1.5. An example of a non-Hausdorff space: a line with a double point.

Definition 1.6. M is called a smooth n-dimensional manifold if M has a countable n-dimensional smooth atlas and M is Hausdorff.

Example 1.7. Atlas for a square in $\mathbb{R}^2$.

Example. Example of smooth manifold: real projective space.

Definition 1.8. Let $U \subseteq \mathbb{R}^n$ be open, $m < n$, and let $f : U \to \mathbb{R}^m$ be a smooth map (i.e., all the partial derivatives are smooth). Let $Df(x) = (\frac{\partial f_i}{\partial x_j})$ be the matrix of partial derivatives at $x \in U$ (differential or Jacobi matrix). Then

(a) $x \in U$ is a regular point of f if $\operatorname{rk} Df(x) = m$ (i.e., $Df(x)$ has a maximal rank);

(b) $y \in \mathbb{R}^m$ is a regular value of f if the full preimage $f^{-1}(y)$ consists of regular points only.

Theorem 1.9 (Corollary of Implicit Function Theorem)**.** *Let $U \subset \mathbb{R}^n$ be open, $f : U \to \mathbb{R}^m$ smooth, $m < n$. If $y \in f(U)$ is a regular value of f then $f^{-1}(y) \subset U \subset \mathbb{R}^n$ is an $(n-m)$-dimensional smooth manifold.*

Examples 1.10–1.11. An ellipsoid as a smooth manifold; matrix groups are smooth manifolds.