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Riemannian Geometry IV, Term 1 (Section 2)

2 Tangent space

Definition 2.1. Let $f : M^m \to N^n$ be a map of smooth manifolds with atlases $(U_i, \varphi_i(U_i), \varphi_i)_{i\in I}$ and $(W_j, \psi_j(W_j), \psi_j)_{j\in J}$. The map f is smooth if it induces smooth maps between open sets in $\mathbb{R}^m$ and $\mathbb{R}^n$, i.e. if $\psi_j \circ f \circ \varphi_i^{-1}\big|_{\varphi_i(U_i\cap f^{-1}(W_j\cap f(U_i)))}$ is smooth for all $i \in I$, $j \in J$.

If f is a bijection and both f and f^{-1} are smooth then f is called a diffeomorphism.

Definition 2.2. A derivation on the set $C^\infty(M, p)$ of all smooth functions on M defined in a neighborhood of p is a linear map $\delta : C^\infty(M, p) \to \mathbb{R}$, s.t. for all $f, g \in C^\infty(M, p)$ holds $\delta(f \cdot g) = f(p)\delta(g) + \delta(f)g(p)$ (the Leibniz rule).

The set of all derivations is denoted by $\mathcal{D}^\infty(M, p)$. This is a real vector space (exercise).

Definition 2.3. The space $\mathcal{D}^\infty(M, p)$ is called the tangent space of M at p, denoted T_pM. Derivations are tangent vectors.

Definition 2.4. Let $\gamma : (a, b) \to M$ be a smooth curve in M, $t_0 \in (a, b)$, $\gamma(t_0) = p$ and $f \in C^\infty(M, p)$. Define the directional derivative $\gamma'(t_0)(f) \in \mathbb{R}$ of f at p along γ by

$$\gamma'(t_0)(f) = \lim_{s\to 0} \frac{f(\gamma(t_0 + s)) - f(\gamma(t_0))}{s} = (f \circ \gamma)'(t_0) = \frac{d}{dt}\bigg|_{t=t_0} (f \circ \gamma)$$

Directional derivatives are derivations (exercise).

Remark. Two curves γ_1 and γ_2 through p may define the same directional derivative.

Notation. Let M^n be a manifold, $\varphi : U \to V \subseteq \mathbb{R}^n$ a chart at $p \in U \subset M$. For $i = 1, \ldots, n$ define the curves $\gamma_i(t) = \varphi^{-1}(\varphi(p) + \boldsymbol{e_i}t)$ for small $t > 0$ (here $\{\boldsymbol{e_i}\}$ is a basis of $\mathbb{R}^n$).

Definition 2.5. Define $\frac{\partial}{\partial x_i}\big|_p = \gamma_i'(0)$, i.e.

$$\frac{\partial}{\partial x_i}\bigg|_p (f) = (f \circ \gamma_i)'(0) = \frac{d}{dt}(f \circ \varphi^{-1})(\varphi(p) + t\boldsymbol{e_i})\big|_{t=0} = \frac{\partial}{\partial x_i}(f \circ \varphi^{-1})(\varphi(p)),$$

where $\frac{\partial}{\partial x_i}$ on the right is just a classical partial derivative.

By definition, we have

$$\left\{\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}\right\} \subseteq \{\text{Directional derivatives}\} \subseteq \mathcal{D}^\infty(M, p)$$

Lemma 2.6. *Let $\gamma : (-\varepsilon, \varepsilon) \to M$ be a smooth curve, $p = \gamma(0)$. Let $\varphi : U \subseteq M \to \mathbb{R}^n$ be a chart with coordinate functions x_i. Then $\gamma'(0)$ is a linear combination of $\left\{\frac{\partial}{\partial x_i}\big|_p\right\}_{1\le i\le n}$.*

Corollary 2.7.

$$\{\textit{Directional derivatives}\} \subseteq \langle \frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n} \rangle \subseteq \mathcal{D}^\infty(M,p).$$

Lemma 2.8. *Let $\varphi : U \subseteq M \to \mathbb{R}^n$ be a chart, $\varphi(p) = 0$. Let $\tilde{\gamma}(t) = (\sum_{i=1}^n k_i \boldsymbol{e_i})\, t : \mathbb{R} \to \mathbb{R}^n$ be a line, where $\{e_1, \dots, e_n\}$ is a basis, $k_i \in \mathbb{R}$. Define $\gamma(t) = \varphi^{-1} \circ \tilde{\gamma}(t) \in M$. Then $\gamma'(0) = \sum_{i=1}^n k_i \frac{\partial}{\partial x_i}$.*

Corollary 2.9.

$$\{\textit{Directional derivatives}\} = \langle \frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n} \rangle \subseteq \mathcal{D}^\infty(M,p).$$

Proposition 2.10.

$$\{\textit{Directional derivatives}\} = \langle \frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n} \rangle = \mathcal{D}^\infty(M,p).$$

Remark. If an n-manifold M is embedded into $\mathbb{R}^N$, then every tangent vector at $p \in M$ can be identified with vector $(\gamma_1'(0), \dots, \gamma_N'(0)) \in \mathbb{R}^N$, where $\gamma : (-\varepsilon, \varepsilon) \to M$ is some smooth curve with $\gamma(0) = p$.

Example 2.11. For the group $SL_n(\mathbb{R}) = \{A \in M_n \mid \det A = 1\}$, the tangent space at I is the set of all trace-free matrices: $T_I(SL_n(\mathbb{R})) = \{X \in M_n(\mathbb{R}) \mid \operatorname{tr} X = 0\}$.

Remark. Since partial derivatives are linearly independent (exercise), the dimension of a tangent space is equal to the dimension of a manifold.

Proposition 2.12. *(Change of basis for T_pM). Let M^n be a smooth manifold, $\varphi_\alpha : U_\alpha \to V_\alpha$ a chart, $(x_1^\alpha, \dots, x_n^\alpha)$ the coordinates in V_α. Let $p \in U_\alpha \cap U_\beta$. Then $\frac{\partial}{\partial x_j^\alpha}\Big|_p = \sum_{i=1}^n \frac{\partial x_i^\beta}{\partial x_j^\alpha} \frac{\partial}{\partial x_i^\alpha}$, where $\frac{\partial x_i^\beta}{\partial x_j^\alpha} = \frac{\partial(\varphi_\beta^i \circ \varphi_\alpha^{-1})}{\partial x_j^\alpha}(\varphi(p))$, $\varphi_\beta^i = \pi_i \circ \varphi_\beta$.*

Definition 2.13. Let M, N be smooth manifolds, let $f : M \to N$ be a smooth map. Define a linear map $Df(p) : T_pM \to T_{f(p)}N$ called the <u>differential</u> of f at p by $Df(p)\gamma'(0) = (f \circ \gamma)'(0)$ for a smooth curve $\gamma \in M$ with $\gamma(0) = p$.

Remark. $Df(p)$ is well defined.

Remark. $Df(p)$ is linear.

Lemma 2.14. *(a) If φ is a chart, then $D\varphi(p) : T_pM \to T_{\varphi(p)}\mathbb{R}^n$ is the identity map taking $\frac{\partial}{\partial x_i}\Big|_p$ to $\frac{\partial}{\partial x_i}$*

(b) For $M \xrightarrow{f} N \xrightarrow{g} L$ we have $D(g \circ f)(p) = Dg(f(p)) \circ Df(p)$.

Example 2.15. Differential of a map from a disc to a sphere.

Tangent bundle and vector fields

Definition 2.16. Let M be a smooth manifold. A disjoint union $TM = \cup_{p \in M} T_pM$ of tangent spaces to each $p \in M$ is called a <u>tangent bundle</u>.

There is a canonical projection $\Pi : TM \to M$, $\Pi(v) = p$ for every $v \in T_pM$.

Proposition 2.17. *The tangent bundle TM has a structure of $2n$-dimensional smooth manifold, s.t. $\Pi : TM \to M$ is a smooth map.*

Definition 2.18. A vector field X on a smooth manifold M is a smooth map $X : M \to TM$ such that $\forall p \in M \ X(p) \in T_pM$

The set of all vector fields on M is denoted by $\mathfrak{X}(M)$.

Remark 2.19.

(a) $\mathfrak{X}(M)$ has a structure of a vector space.

(b) Vector fields can be multiplied by smooth functions.

(c) Taking a coordinate chart $(U, \varphi = (x_1, \dots, x_n))$, any vector field X can be written in U as $X(p) = \sum_{i=1}^n f_i(p) \frac{\partial}{\partial x_i} \in T_pM$, where $\{f_i\}$ are some smooth functions on U.

Examples 2.20–2.21. Vector fields on $\mathbb{R}^2$ and 2-sphere.

Remark 2.22. Observe that for $X \in \mathfrak{X}(M)$ we have $X(p) \in T_pM$, i.e. $X(p)$ is a directional derivative at $p \in M$. Thus, we can use the vector field to differentiate a function $f \in C^\infty(M)$ by $(Xf)(p) = X(p)f$, so that we get another smooth function $Xf \in C^\infty(M)$.

Proposition 2.23. *Let $X, Y \in \mathfrak{X}(M)$. Then there exists a unique vector field $Z \in \mathfrak{X}(M)$ such that $Z(f) = X(Y(f)) - Y(X(f))$ for all $f \in C^\infty(M)$.*

This vector field $Z = XY - YX$ is denoted by $[X, Y]$ and called the Lie bracket of X and Y.

Proposition 2.24. *Properties of Lie bracket:*

(a) $[X, Y] = -[Y, X]$*;*

(b) $[aX + bY, Z] = a[X, Z] + b[Y, Z]$ *for* $a, b \in \mathbb{R}$*;*

(c) $[[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] = 0$ *(Jacobi identity);*

(d) $[fX, gY] = fg[X, Y] + f(Xg)Y - g(Yf)X$ *for* $f, g \in C^\infty(M)$*.*

Definition 2.25. A Lie algebra is a vector space $\mathfrak{g}$ with a binary operation $[\cdot, \cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ called the Lie bracket which satisfies first three properties from Proposition 2.24.

Proposition 2.24 implies that $\mathfrak{X}(M)$ is a Lie algebra.

Theorem 2.26 (The Hairy Ball Theorem)**.** *There is no non-vanishing continuous vector field on an even-dimensional sphere S^{2m}.*

Corollary. Let $f : S^{2m} \to S^{2m}$ be a continuous map, and suppose that for any $p \in S^{2m}$ we have $f(p) \neq -p$. Then f has a fixed point, i.e. there exists $q \in S^{2m}$ s.t. $f(q) = q$.