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Riemannian Geometry IV, Term 1 (Section 3)

3 Riemannian metric

Definition 3.1. Let M be a smooth manifold. A Riemannian metric $g_p(\cdot,\cdot)$ or $\langle\cdot,\cdot\rangle_p$ is a family of real inner products $g_p : T_pM \times T_pM \to \mathbb{R}$ depending smoothly on $p \in M$. A smooth manifold M with a Riemannian metric g is called a Riemannian manifold (M, g).

Examples 3.2–3.3. Euclidean metric on $\mathbb{R}^n$, induced metric on $M \subset \mathbb{R}^n$.

Definition 3.4. Let (M, g) be a Riemannian manifold. For $v \in T_pM$ define the length of v by $0 \le \|v\|_g = \sqrt{g_p(v, v)}$.

Example 3.5. Three models of hyperbolic geometry:

model	notation	M	g
Hyperboloid	$\mathbb{W}^n$	$\{y \in \mathbb{R}^{n+1} \mid q(y,y) = -1, y_{n+1} > 0\}$ where $q(x,y) = \sum_{i=1}^n x_iy_i - x_{n+1}y_{n+1}$	$g_x(v,w) = q(v,w)$
Poincaré ball	$\mathbb{B}^n$	$\{x \in \mathbb{R}^n \mid \|x\|^2 = \sum\limits_{i=1}^n x_i^2 < 1\}$	$g_x(v,w) = \frac{4}{(1-\|x\|^2)^2}\langle v,w\rangle_{\mathrm{Eucl}}$
Upper half-space	$\mathbb{H}^n$	$\{x \in \mathbb{R}^n \mid x_n > 0\}$	$g_x(v,w) = \frac{1}{x_n^2}\langle v,w\rangle_{\mathrm{Eucl}}$

Definition 3.6. Given two vector spaces V_1, V_2 with real inner products $(V_i, \langle\cdot,\cdot\rangle_i)$, an isomorphism $T : V_1 \to V_2$ of vector spaces is a linear isometry if $\langle Tv, Tw\rangle_2 = \langle v, w\rangle_1$ for all $v, w \in V_1$.

This is equivalent to preserving the lengths of all vectors (since $\langle v, w\rangle = \frac{1}{2}(\langle v + w, v + w\rangle - \langle v, v\rangle - \langle w, w\rangle)$).

Definition 3.7. A diffeomorphism $f : (M, g) \to (N, h)$ of two Riemannian manifolds is an isometry if $Df(p) : T_pM \to T_{f(p)}N$ is a linear isometry for all $p \in M$.

Theorem 3.8 (Nash embedding theorem)**.** *For any Riemannian manifold (M^m, g) the exists an isometric embedding into $\mathbb{R}^k$ for some $k \in \mathbb{N}$. If M is compact, there exists such $k \le \frac{m(3m+1)}{2}$, and if M is not compact, there is such $k \le \frac{m(m+1)(3m+1)}{2}$.*

Definition 3.9. (M, g) is a Riemannian manifold, $c : [a, b] \to M$ is a smooth curve. The length $L(c)$ of c is defined by $L(c) = \int_a^b \|c'(t)\|dt$, where $\|c'(t)\| = \langle c'(t), c'(t)\rangle_{c(t)}^{1/2}$. The length of a piecewise-smooth curve is defined as the sum of lengths of its smooth pieces.

Theorem 3.10 (Reparametrization)**.** *Let $\varphi : [c, d] \to [a, b]$ be a strictly monotonic smooth function, $\varphi' \neq 0$, and let $\gamma : [a, b] \to M$ be a smooth curve. Then for $\tilde{\gamma} = \gamma \circ \varphi : [c, d] \to M$ holds $L(\gamma) = L(\tilde{\gamma})$.*

Definition 3.11. A smooth curve $c : [a, b] \to M$ is arc-length parametrized if $\|c'(t)\| \equiv 1$.

Proposition 3.12 (evident)**.** *If a curve $c : [a, b] \to M$ is arc-length parametrized, then $L(c) = b - a$.*

Proposition 3.13. *Every curve has an arc-length parametrization.*

Example 3.14. Length of vertical segments in $\mathbb{H}$. Shortest paths between points on vertical rays.

Definition 3.15. Define a <u>distance</u> $d : M \times M \to [0, \infty)$ on (M, g) by $d(p, q) = \inf_\gamma \{L(\gamma)\}$, where γ is a piecewise smooth curve connecting p and q.

Remark. (M, d) is a metric space.

Example. Example of a manifold with non-complete Riemannian metric.

Example 3.16. Induced metric on $S^1 \subset \mathbb{R}^2$.

Definition 3.17. If (M, g) is a Riemannian manifold, then any subset $A \subset M$ is also a metric space with the <u>induced metric</u> $d|_{A\times A} : A \times A \to [0, \infty)$ defined by $d(p, q) = \inf_\gamma \{L(\gamma) \mid \gamma : [a, b] \to A, \gamma(a) = p, \gamma(b) = q\}$, where the length $L(\gamma)$ is computed in M.

Example 3.18. Punctured sphere: $\mathbb{R}^n$ with metric $g_x(v, w) = \frac{4}{(1+\|x\|^2)^2} \langle v, w\rangle_{\text{Eucl}}$.