Durham University
Pavel Tumarkin

Michaelmas 2022

Topics in Combinatorics IV, Homework 1 (Week 1)

Due date for starred problems: **Friday, October 21, 6pm**.

1.1. Compute the number of Dyck paths of length $2n$ which start with two steps up.

1.2. ($\star$)

(a) Let P be a Dyck path of length $2n$, let $a_1, \dots, a_n$ be the positions of the steps "down", $1 \le a_i \le 2n$. Show that $a_i \ge 2i$ for every $i \le n$.

(b) Show that the number of strictly increasing sequences $(a_1, \dots, a_n)$ of integers satisfying $2i \le a_i \le 2n$ is the n-th Catalan number C_n.

(c) Show that the number of strictly increasing sequences $(a_1, \dots, a_{n-1})$ of integers satisfying $1 \le a_i \le 2i$ is the n-th Catalan number C_n.

1.3. ($\star$) Show explicitly that the number of triangulations of an $(n+2)$-gon satisfies the Catalan recursion (see Lemma 1.10 from lectures).

1.4. Find a bijection between ballot sequences of length $2n$ and bracketings of $n+1$ variables. *Hint*: assign to every $+1$ in the sequence an opening bracket.

1.5. Given a ballot sequence $\varepsilon_1, \dots, \varepsilon_{2n}$, one can write a sequence of differences $a_i = \varepsilon_{i+1} - \varepsilon_i$, $1 \le i \le 2n-1$. Characterize all such sequences (and thus, get another definition of Catalan numbers).