Durham University
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Topics in Combinatorics IV, Homework 5 (Week 5)

Due date for starred problems: **Friday, November 18, 6pm**.

5.1. Let c_n denote the number of c-objects on n labeled nodes (as in the lectures), $n \geq 1$. Denote by $d_{n,k}$ the number of d-objects on n nodes with k components, i.e. the number of collections of k c-objects with total number of nodes being n (e.g., $d_{n,1} = c_n$, and $\sum_k d_{n,k} = d_n$). Define

$$d(x, y) = \sum_{n\geq 0} \sum_{k\geq 0} d_{n,k} \frac{x^n}{n!} y^k$$

Show that $d(x, y) = e^{y\cdot c(x)}$, where $c(x)$ is the exponential generating function of (c_n).

5.2. Recall that Stirling number of second kind $S(n, k)$ is defined as the number of set partitions of $[n]$ into k blocks.

(a) Show that $\sum_{n,k\geq 0} S(n, k) \frac{x^n}{n!} y^k = e^{y(e^x-1)}$.

(b) Prove the following recurrence relation:

$$S(n, k) = S(n - 1, k - 1) + kS(n - 1, k)$$

5.3. ($\star$) Define the *falling factorial* $y_{(k)} = y(y - 1) \dots (y - k + 1) = k!\binom{y}{k}$ for any $y \in \mathbb{R}$.

(a) Show that the number of surjective functions $f : [n] \to [k]$ is equal to $S(n, k) \cdot k!$.

(b) Show that for any $m, n \in \mathbb{N}$

$$\sum_{k=0}^{n} \binom{m}{k} S(n, k) \cdot k! = m^n$$

(c) Show that

$$\sum_{k=0}^{n} S(n, k) y_{(k)} = y^n$$

5.4. Define the *signless Stirling number of the first kind* $c(n, k)$ as the number of permutations $w \in S_n$ with $\operatorname{cyc}(w) = k$, and *Stirling number of the first kind* as $s(n, k) = (-1)^{n-k} c(n, k)$. We define $c(0, 0) = 1$ and $c(n, 0) = c(0, n) = 0$ for $n > 0$.

(a) Show that $c(n, k) = c(n - 1, k - 1) + (n - 1)c(n - 1, k)$.

(b) Define the *raising factorial* $y^{(k)} = y(y + 1) \dots (y + k - 1) = k!\binom{y+k-1}{k}$ for any $y \in \mathbb{R}$. Show that

$$\sum_{k=0}^{n} c(n, k) x^k = x^{(n)}$$

(c) Show that

$$\sum_{k=0}^{n} s(n, k) x^k = x_{(n)}$$