Durham University
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Topics in Combinatorics IV, Homework 6 (Week 6)

Due date for starred problems: **Friday, November 18, 6pm**.

6.1. Recall that given $w = w_1 w_2 \dots w_n \in S_n$, $\operatorname{inv}(w)$ is the number of inversions (i.e. pairs $i < j$ such that $w_i > w_j$), $\operatorname{des}(w)$ is the number of descents (i.e. places $i \in [n-1]$ such that $w_i > w_{i+1}$), and $\operatorname{maj}(w)$ is the sum of all $i \in [n-1]$ such that i is a descent of w.

Show that two statistics maj and inv are equidistributed.

Hint: use induction.

6.2. Recall that given $w \in S_n$, $\operatorname{exc}(w)$ is the number of excedances of w (i.e. places $i \in [n]$ such that $i < w_i$).

Complete the proof of Theorem 3.13: show that statistics des and exc are equidistributed.

6.3. Let $w = w_1 w_2 \dots w_n \in S_n$, $n \geq 2$. $i \in [n]$ is a *weak excedance* of w if $w_i \geq i$. Denote by $\operatorname{wexc}(w)$ the number of weak excedances of $w \in S_n$.

Show that statistics exc and $\operatorname{wexc} - 1$ are equidistributed.

6.4. ($\star$) Define *Eulerian numbers* $A(n, k)$ as the numbers of permutations $w \in S_n$ with $\operatorname{des}(w) = k - 1$, $k \leq n$.

Show that $A(n, k+1) = (n-k)A(n-1, k) + (k+1)A(n-1, k+1)$.