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Topics in Combinatorics IV, Problems Class 1 (Week 2)

1.1. Construct a bijection between Dyck paths of length $2n$ and plane trees with $n+1$ vertices.

Take a tree, draw a small neiborhood of it, then the boundary of the neighborhood is a closed loop. Let us move along this boundary clockwise starting from the point just above the root. Now we associate to the tree a Dyck path as follows: if we are passing an edge by downwards, we make a step up, and otherwise we make a step down. See Fig. 1.1 for an example.

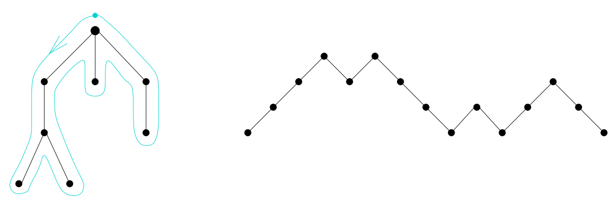

Figure 1.1: Construction of a Dyck path by a plane tree

1.2. Construct a bijection between triangulations of an $(n+2)$-gon and complete binary trees with $n+1$ leaves.

Given a polygon, mark one side of it. Now consider a triangulation. Put a vertex in every triangle and just outside every side of the polygon except for the one marked. Now connect vetrices by edges if they intersect precisely one side or diagonal (we get a *dual graph* of the triangulation without one vertex). We got a complete binary tree, where the root is the vertex in the triangle with marked side.

See Fig. 1.2 for an example.

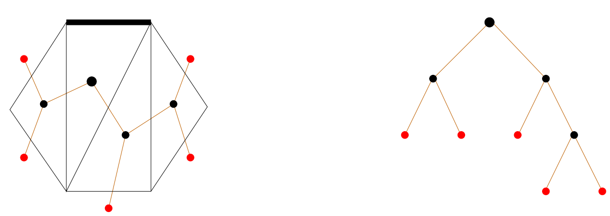

Figure 1.2: Construction of a complete binary tree by a triangulation

1.3. For any sequence $(\varepsilon_1, \dots, \varepsilon_{2n+1})$, $\varepsilon_i = \pm 1$, adding up to -1, consider all its $2n+1$ *cyclic shifts*:

$$(\varepsilon_1, \dots, \varepsilon_{2n+1}),\ (\varepsilon_2, \varepsilon_3, \dots, \varepsilon_{2n+1}, \varepsilon_1),\ (\varepsilon_3, \dots, \varepsilon_{2n+1}, \varepsilon_1, \varepsilon_2), \dots, (\varepsilon_{2n+1}, \varepsilon_1, \dots, \varepsilon_{2n}).$$

Call a sequence *good* if $\sum_{i=1}^{k} \varepsilon_i \geq 0$ for every $k < 2n+1$.

(a) Show that all cyclic shifts are distinct.

(b) Show that out of $2n+1$ shifts of a sequence precisely one is good.

The main observation is the following. Consider the leftmost lowest point of the Dyck path, assume that its x-coordinate is k. Then the x-coordinate of the leftmost lowest point of the i-th shift is $k-i$ (considered modulo $2n+1$).

Indeed, we need to check this for shift by one only. k is the leftmost lowest point iff $\sum_{j=1}^{m} \varepsilon_j > \sum_{j=1}^{k} \varepsilon_j$ for any $m < k$ and $\sum_{j=1}^{r} \varepsilon_j \geq \sum_{j=1}^{k} \varepsilon_j$ for any $r > k$. Assuming $k \neq 1$, after shifting by one the y-coordinate of a point $m-1$ is equal to $\sum_{j=2}^{m} \varepsilon_j$, and since we have $\sum_{j=2}^{m} \varepsilon_j > \sum_{j=2}^{k} \varepsilon_j$ for any $m < k$, all points on the left of $k-1$ have bigger y-coordinate (and, in particular, y-coordinate of $k-1$ is negative). On the right, we also have $\sum_{j=2}^{r} \varepsilon_j \geq \sum_{j=2}^{k} \varepsilon_j$ for any $k < r \leq 2n+1$. The point $2n+1$ has y-coordinate -1, so it is not lower than $k-1$ either.

If $k = 1$, then $\varepsilon_1 = -1$, and for every m we have $\sum_{j=1}^{m} \varepsilon_j \geq -1$. Thus, $\sum_{j=2}^{m} \varepsilon_j \geq 0$ for every $m \leq 2n+1$, and the only point with negative y-coordinate in the shift is $2n+1$.

The observation proves both (a) and (b): every shift has different leftmost lowest point, and precisely one of them has it at point $2n+1$.