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Topics in Combinatorics IV, Problems Class 3 (Week 6)

The class was devoted to the second assignment questions 3.2 and 4.2, in particular solutions alternative to the ones published (see solutions uploaded), some of which are shown below.

3.2 (a) Note that at every step the sequence becomes longer precisely by the number of edges of the minimal polygon containing the edge e minus one, so we can establish a bijection between the entries of the sequence and all segments in the dissection (except the edge e itself). Thus, the length of the sequence is $(n+2)+d-1=n+d+1$.

(c) (courtesy of Heath Pearson)

First, construct a bijection between dissections and those plane rooted trees having no vertices with precisely one child. The bijection is similar to the one which was constructed in Problems Class 1 (week 2) between triangulations and complete plane binary trees, just the condition of being binary is dropped (as a domain may now have more than three sides). Now, write at every vertex the number of children minus one (i.e., leaves will be assigned -1 each), and walk around the tree clockwise, writing down every new number met as the next entry of the sequence. The last -1 should be omitted. The resulting sequence is precisely the required sequence (that can also be obtained by the procedure described in Exercise 3.2). The construction shows the way to reconstruct (uniquely) a tree by a sequence, so there is a bijection between trees and sequences, which implies that there is a bijection between dissections and sequences.

4.2 We want to construct a bijection between Dyck paths of length $2n$ with k peaks and non-crossing partitions of $[n]$ with k blocks.

Take a Dyck path, and index all steps up in turn. Now for each step up find the corresponding step down and assign to it the same number. To do this observe that to a Dyck path we can associate a sequence of n opening and n closing brackets (up – opening, down – closing), such that the number of opening brackets at every moment is not less the number of closing ones; then every opening bracket has the corresponding closing one, this will correspond precisely to the matching down step).

Now every block of the partition is the indices of consecutive steps down. Since the number of peaks is equal to k, the number of blocks is also equal to k.

The easiest way to prove that the partition is non-crossing is by induction. If the Dyck path reaches the x-axis at $2m < 2n$, then it is a union of two smaller Dyck paths, and by the induction assumption our partition is a union of two non-crossing partitions of $[m]$ and $\{m+1,\dots,n\}$. If the Dyck path is always above x-axis, then observe that 1 is in the same block with n, and by removing 1 we get a shorter Dyck path which, by the induction assumption, corresponds to a non-crossing partition of $\{2,\dots,n\}$. We need to prove that if we add 1 to the block containing n, the partition will still be non-crossing. Indeed, if it is not, then there are arc $(1, j)$ and (i, m) for some $1 < i < j < m < n$ in the corresponding arc diagram. But note that j is in the same block with n, and thus there is a sequence of arcs $(j = j_0, j_1), (j_1, j_2) \dots (j_s, j_{s+1} = n)$. Then at least one of this arcs intersects the arc (i, m), which contradicts the fact the partition of $\{2, \dots, n\}$ was non-crossing.

It is easy to see that different Dyck paths give rise to different partitions, so the map is injective. Since the total number of paths and total number of partitions are the same (i.e., C_n), the map is also surjective, and thus a bijection.