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Topics in Combinatorics IV, Problems Class 4 (Week 8)

The first part of the class was devoted to the third assignment question 5.3.

4.1. Let L be a finite lattice. Define P to be the poset of all join-irreducible elements of L (with the order inherited from L). Let $x \in L$, consider $I_x = \{t \in P \mid t \le x\} \subset P$. Show that $x = \vee\{t \mid t \in I_x\}$.

Let $y = \vee\{t \mid t \in I_x\}$, and assume $y \neq x$. Since $x \ge t$ for all $t \in I_x$, we have $x \ge y$. Observe that $y \ge t$ for all $t \in I_x$.

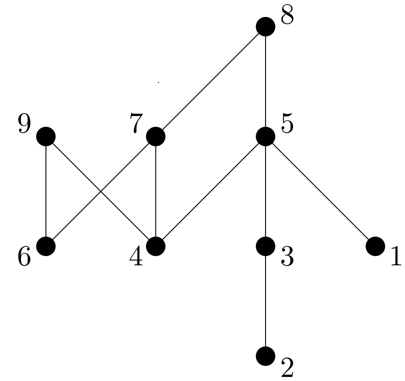
Define a set $X = \{z \in L \mid z \le x, z \not\le y\}$. Note that $X \neq \emptyset$ since $x \in X$. As L is finite, we can take a minimal element of X, call it x_0. Observe that $x_0 \neq \hat{0}$ as $\hat{0} \le y$, so there exists at least one element which is less than x_0. Then we have a dihotomy: either x_0 is join-irreducible or not.

Assume first that x_0 is join-irreducible. Since $x_0 \in X$, $x_0 \le x$, and thus $x_0 \in I_x$, which implies $x_0 \le y$, which condradicts $x_0 \in X$.

Now assume that x_0 is not join-irreducible, so $x_0 = a \vee b$, $a, b < x_0$. Since x_0 is minimal, both $a, b \notin X$. Note that $a, b < x_0 \le x$, which implies that $a, b \le y$ (otherwise they would have lied in X). So, we have $a, b \le y$, which implies $a \vee b \le y$. But $a \vee b = x_0 \in X$, so $a \vee b \not\le y$, and we came to a contradiction again.

4.2. Given $w \in S_n$, define a poset P_w as follows: elements of P_w are elements of $[n]$, and $w_i <_{P_w} w_j$ if $w_i < w_j$ and $i < j$. Now, construct a Young diagram $\lambda(P_w)$ as in the Greene's Theorem. Check that for $w = 6\,4\,9\,7\,2\,3\,1\,5\,8 \in S_9$ the Young diagram $\lambda(P_w)$ coincides with the Schensted shape of λ.

To construct $\lambda(P_w)$, we need first to draw the Hasse diagram of P_w:



Now we compute numbers l_i, where l_i is the maximal size of a union of i chains.

Clearly, the only longest chain is $2 \lessdot 3 \lessdot 5 \lessdot 8$, so $l_1 = 4$.

The next number, l_2, cannot be equal 7 as for that one should have disjoint chains of size 3 and 4, but after removing the longest chain the maximal chain left has size 2. Therefore, $l_2 = 6$ (add e.g. chain $6 \lessdot 9$).

Now, l_3 cannot be equal to 9, as this would imply that we covered the whole P_w by 3 chains, which is impossible as there are four minimal elements. At the same time, we can add chain $4 \lessdot 7$ to see that $l_3 = 8$.

Finally, $l_4 = 9$ (just add 1).

Therefore, we have $\lambda_1 = l_1 = 4$, $\lambda_2 = l_2 - l_1 = 6 - 4 = 2$, $\lambda_3 = l_3 - l_2 = 8 - 6 = 2$, $\lambda_4 = l_4 - l_3 = 9 - 8 = 1$, so $\lambda(p_w) = (4, 2, 2, 1) \vdash 9$.

To compute the insertion tableau of $w = 6\,4\,9\,7\,2\,3\,1\,5\,8$ we apply the RSK algorithm.

Step 1: 6 is inserted in the box $(1, 1)$.

Step 2: 4 is inserted in the box $(1, 1)$, and 6 is pushed down into the box $(1, 2)$.

Step 3: 9 is inserted in the box $(1, 2)$.

Step 4: 7 is inserted in the box $(1, 2)$, and thus pushes down 9 into the second row, where it goes to the box $(2, 2)$.

Step 5: 2 is inserted in the box $(1, 1)$, and thus pushes down 4 into the second row, where it goes to the box $(2, 1)$ and pushes 6 down into a new box $(3, 1)$.

Step 6: 3 is inserted in the box $(1, 2)$, and thus pushes down 7 into the second row, where it goes to the box $(2, 2)$ and pushes 6 down into the third row, where it forms a new box $(3, 2)$.

Step 7: 1 is inserted in the box $(1, 1)$, it pushes down 2 to the box $(2, 1)$, which pushes 4 to the box $(3, 1)$, and thus 6 is pushed down into the fourth row to form a new box $(4, 1)$.

Step 8: 5 is inserted in the box $(1, 3)$.

Step 9: 8 is inserted in the box $(1, 4)$.

As a result, we get the following SYT which is of shape $(4, 2, 2, 1)$ as required.

$$P = \begin{array}{|c|c|c|c|} \hline 1 & 3 & 5 & 8 \\ \hline 2 & 7 \\ \cline{1-2} 4 & 9 \\ \cline{1-2} 6 \\ \cline{1-1} \end{array}$$