Jacobs University
School of Engineering and Science

Spring Term 2009

ESM 2B, Homework 11

Due Date: 14:00 Wednesday, May 13.

Explain your answers! Problems marked ($\star$) are bonus ones.

11.1. Prove the following identities:

(a) $\binom{n}{k-1} + \binom{n}{k} = \binom{n+1}{k}$;

(b) $\sum_{k=0}^{n} \binom{n}{k} = 2^n$;

(c)($\star$) $\sum_{i=k}^{n} \binom{i}{k} = \binom{n+1}{k+1}$;

11.2. A boy is selected at random from among the children belonging to families with n children. What is the probability that the boy has $k-1$ brothers?

11.3. Suppose we have a box with 10 balls, 6 of which are black and 4 are white. Find the probability of drawing 2 white balls from the box if:

(a) the first ball is returned into the box before the second ball is drawn;

(b) the first ball is put aside after being drawn.

11.4. Suppose one in 151 people has a certain disease. There is a test for this disease that is completely reliable in detecting the disease but gives a false positive result on 1% of the noninfected population. Given that somebody is tested positively, what is the probability that this person indeed has the disease?
Hint: Consider events $A =${Test is positive} and $B =${Person is infected}, and then use Bayes' rule.

11.5. Assume Peter leaves his office every day at a random time, goes to the bus stop, from where every 10 minutes leaves a bus.

(a) Determine the expected waiting time.

(b) After four days, Peter observed that he never waited longer than 8 minutes. What is the probability of this event?

(c) What is the probability that Peter waited at least one time longer than 8 minutes in 4 days?

11.6. The moment generating function of a certain random variable X is

$$M_X(t) = \frac{c}{c-t}$$

where $c > 0$ is a parameter. Determine the expectation and variance of X.

11.7. Suppose a random number generator produces uniformly distributed numbers X in the interval $[0, 4]$, and another random number generator produces uniformly distributed numbers Y in the interval $[2, 5]$. What is the probability that $X + Y \geq 3$?

11.8. ($\star$) Let X and Y be independent random variables that take non-negative real values and whose associated density functions are $f_X(t) = \lambda e^{-\lambda t}$, $f_Y(t) = \mu e^{-\mu t}$ for $t > 0$, and $f_X(t) = f_Y(t) = 0$ for $t < 0$. Compute the probability $P(a < X + Y < b)$ for real $a < b$.