Jacobs University
School of Engineering and Science

Spring Term 2010

ESM 2B, Homework 11

Due Date: 17:00 Wednesday, May 12.

Explain your answers! Problems marked ($\star$) are bonus ones.

11.1. A bag contains 5 white and 3 red balls. 4 balls are drawn simultaneously at random from the bag. Let X be the number of white balls drawn.

(a) Write down the probability function of X.

(b) Compute expectation and variance of X.

11.2. Let X_1, X_2, ... be independent discrete random variables taking values in $\{-\frac{1}{2}, \frac{1}{2}\}$ with probability function

$$p(-\tfrac{1}{2}) = q\,, \qquad p(\tfrac{1}{2}) = (1-q)\,,$$

and $p(x) = 0$ for all other values of x.

(a) Find the expectation for the random variable $Z_N = \frac{X_1+\cdots+X_N}{N}$ as $N \to \infty$.

(b) Let $Y_n = 2^{-n}X_n$. Find the expectation for the random variable $Z_N = Y_1 + \cdots + Y_N$ as $N \to \infty$.

11.3. Suppose one in 151 people has a certain disease. There is a test for this disease that is completely reliable in detecting the disease but gives a false positive result on 1% of the noninfected population. Given that somebody is tested positively, what is the probability that this person indeed has the disease?
Hint: Consider events A ={Test is positive} and B ={Person is infected}, and then use Bayes' rule.

11.4. Assume Peter leaves his office every day at a random time, goes to the bus stop, from where every 8 minutes leaves a bus.

(a) Determine the expected waiting time.

(b) After four days, Peter observed that he never waited longer than 6 minutes. What is the probability of this event?

(c) What is the probability that Peter waited at least one time longer than 5 minutes in 4 days?

11.5. The moment generating function of a certain random variable X is $M_X(t) = \dfrac{c}{c-t}$, where $c > 0$ is a parameter. Determine the expectation and variance of X.

11.6. Suppose a random number generator produces uniformly distributed numbers X in the interval $[0, 3]$, and another random number generator produces uniformly distributed numbers Y in the interval $[1, 5]$. What is the probability that $X + Y \geq 3$?

11.7. ($\star$) Let X and Y be independent random variables that take non-negative real values and whose associated density functions are $f_X(t) = \lambda e^{-\lambda t}$, $f_Y(t) = \mu e^{-\mu t}$ for $t > 0$, and $f_X(t) = f_Y(t) = 0$ for $t < 0$. Compute the probability $P(a < X + Y < b)$ for real $a < b$.

11.8. ($\star$) A lighthouse is situated at a distance L from a straight coastline, opposite to a point O, and sends out a narrow continuous beam of light simultaneously in opposite directions. The beam rotates with a constant angular velocity. If the random variable Y is the distance along the coastline, measured from O, of the spot that the light beam illuminates, find its probability density function.