Jacobs University
School of Engineering and Science

Spring Term 2010

ESM 2B, Homework 2

Due Date: 14:00 Thursday, February 25.

Explain your answers! Problems marked ($\star$) are bonus ones.

2.1. Find the kernel and the image of the following linear maps:

(a) $\mathcal{A} : \mathbb{R}^n \to \mathbb{R}^n$, $\mathcal{A}v = 0$;

(b) $\mathcal{A} : \mathbb{C}^n \to \mathbb{C}^n$, $\mathcal{A}v = v$;

(c) $\mathcal{A} : \mathbb{R}^3 \to \mathbb{R}^3$, $\mathcal{A}(x, y, z) = (y, z, 0)$;

(d) $\mathcal{A} : \mathbb{R}^4 \to \mathbb{R}^3$, $\mathcal{A}(x, y, z, t) = (x - y, y - z, z - t)$;

($\star$) $\mathcal{A} : \mathbb{R}[x] \to \mathbb{R}[x]$, $(\mathcal{A}p)(x) = p(\alpha x^2 - \beta)$, where $\alpha, \beta \in \mathbb{R}$ are fixed numbers.

2.2. Compute matrices of linear maps from Problem 1 in some bases. Which of the matrices do not depend on bases?

2.3. Let V be the space of polynomials with real coefficients of degree at most 2, and $\alpha \in \mathbb{R}$ is a fixed number, $\alpha \neq 0$. Define two maps $\mathcal{A}, \mathcal{B} : V \to V$ by

$$(\mathcal{A}p)(x) = p(\alpha x); \qquad (\mathcal{B}p)(x) = p(x - \alpha).$$

Show that $\mathcal{A}, \mathcal{B}$ are linear operators, and compute the matrices of $\mathcal{A}$ and $\mathcal{B}$ in the basis $\{1, x, x^2\}$.

2.4. Define $A = \begin{pmatrix} 2 & 0 & 1 \\ -1 & 3 & 1 \\ 0 & -2 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 2 & 0 \\ -1 & 2 \\ 2 & 3 \end{pmatrix}$, $C = \begin{pmatrix} 2 & 1 & 1 \end{pmatrix}$. Compute the following products if possible:

(a) AB; (b) BA; (c) CA; (d) CC^T; (e) C^TC; (f) C^2; (g) A^2; (h) BB^T; (i) AC; (j) AC^T; (k) CB; (l) CAC^T; (m) $C^T(C^TB)$;

2.5. (a) Find all the matrices $X \in \mathrm{M}_2$ commuting with $A = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$.

($\star$) Find all the matrices $X \in \mathrm{M}_n$ commuting with $A = \begin{pmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \lambda_n \end{pmatrix}$.

2.6. Solve the following equation with respect $\{x, y, z\}$:

$$\begin{pmatrix} x & y \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 4 & 0 \\ y & 2 \end{pmatrix}^T + \begin{pmatrix} 0 & -xy \\ 2 & -2 \end{pmatrix} = \begin{pmatrix} 4 & 2 \\ z & -1 \end{pmatrix}$$