Jacobs University
School of Engineering and Science

Spring Term 2010

ESM 2B, Homework 4

Due Date: 14:00 Thursday, March 11.

Explain your answers! Problems marked ($\star$) are bonus ones.

4.1. Compute the characteristic polynomial and find the eigenvalues (with multiplicities), eigenvectors, and bases of eigenspaces for the following matrices. Which of them are diagonalizable?

$$(a)\ \begin{pmatrix} 1 & 1 \\ -1 & 3 \end{pmatrix} \qquad (b)\ \begin{pmatrix} 0 & 1 & -1 \\ 2 & 4 & 1 \\ 1 & 0 & 3 \end{pmatrix} \qquad (c)\ \begin{pmatrix} 3 & 3 & 0 \\ 0 & 1 & 1 \\ 0 & -1 & 1 \end{pmatrix}$$

$$(d)\ \begin{pmatrix} 0 & 1 & 1 \\ -4 & -4 & 0 \\ 0 & 1 & 3 \end{pmatrix} \qquad (e)\ \begin{pmatrix} 4 & -3 & 3 \\ -2 & 2 & -2 \\ -8 & 7 & -7 \end{pmatrix}$$

4.2. Show that

(a) the product of eigenvalues of A equals $\det \mathrm{A}$;

(b)($\star$) the sum of eigenvalues of A equals $\mathrm{tr}\, \mathrm{A}$.

4.3. Decide which of the following matrices are diagonalizable. If so, find a diagonal matrix D and an invertible matrix C such that $A = CDC^{-1}$.

$$(a)\ A = \begin{pmatrix} 0 & 3 \\ 1 & 5 \end{pmatrix} \qquad (b)\ A = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix} \qquad (c)\ A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad (d)\ A = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 3 & 0 \\ -2 & 0 & 2 \end{pmatrix}$$

4.4. Let $x_1, \dots, x_n$ be complex numbers.

(a) Compute the following determinant

$$\begin{vmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ x_1^2 & x_2^2 & x_3^2 \end{vmatrix}$$

(b)($\star$) Prove the following equality (*Vandermonde determinant*):

$$\begin{vmatrix} 1 & 1 & 1 & \dots & 1 \\ x_1 & x_2 & x_3 & \dots & x_n \\ x_1^2 & x_2^2 & x_3^2 & \dots & x_n^2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ x_1^{n-1} & x_2^{n-1} & x_3^{n-1} & \dots & x_n^{n-1} \end{vmatrix} = \prod_{1 \le i < j \le n} (x_j - x_i)$$

Hint: use *Bezout Theorem*: if a is a root of a polynomial $P(x)$, then $P(x) = (x - a)P_1(x)$ for some polynomial $P_1(x)$.

(c)($\star$) Show that eigenvectors of matrix $A \in \mathrm{Mat}_n$ with mutually distinct eigenvalues are linearly independent.