Jacobs University
School of Engineering and Science

Spring Term 2010

ESM 2B, Homework 8

Due Date: 14:00 Thursday, April 22.

Explain your answers! Problems marked ($\star$) are bonus ones.

8.1. Find the Fourier transform of the function $f(x) = e^{-\alpha x^2}$

(use the fact that $\int\limits_{-\infty}^{\infty} e^{-x^2}\, dx = \sqrt{\pi}$).

8.2. Assuming the Fourier transform of a function $f(x)$ to be $\hat{f}(y)$, compute the Fourier transform of

(a) $g(x) = f(x-a)$, $a \in \mathbb{R}$; (b) $g(x) = f(x/a)$, $a > 0$.

8.3. Consider the following equation

$$\frac{d^2u}{dx^2} - u(x) = f(x)$$

with respect to u. Show that the solution $u(x)$ can be written as

$$u(x) = \frac{-1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{i\xi x}\, \hat{f}(\xi)}{1+\xi^2}\, d\xi$$

where $\hat{f}(\xi)$ is the Fourier transform of $f(x)$.

8.4. Let $f * g$ be a convolution of two functions. Show that

(a) $f * g = g * f$; (b) $(f * g) * h = f * (g * h)$; (c) $f * (g + h) = f * g + f * h$.

(d) Is it true that $(f * g)h = f * (gh)$?

8.5. Let $\delta(x)$ be the δ-function. Compute

(a) $\int\limits_{-2}^{2} \delta(x)(2 - 2x^2 + e^x)\, dx$; (b) $\int\limits_{-2}^{2} \delta(x+1)(1 + 2x^3 - \cos^3(\pi x))\, dx$;

(c) $\int\limits_{-2}^{2} \delta(x-3)e^{-2x^2} \cos x\, dx$; (d)($\star$) $\int\limits_{-2\pi}^{2\pi} \delta(x^2 - \pi^2) \cos x\, dx$.

8.6. ($\star$) One can define (so-called distributional) derivatives of the δ-function via

$$\int_{-\infty}^{\infty} f(x)\, \delta^{(n)}(x)\, dx = (-1)^n\, f^{(n)}(0)$$

for any n times differentiable function $f(x)$.

So we may be tempted to write a Taylor series for the δ-function,

$$\delta(x+a) = \sum_{n=0}^{\infty} \frac{\delta^{(n)}(x)}{n!}\, a^n\, .$$

It appears that the left side is zero except at $x = -a$, while the right side is zero except at $x = 0$. Resolve this paradox.