Jacobs University
School of Engineering and Science

Spring Term 2010

Linear Algebra II, Homework 3

Due Date: Thursday, February 25, in class.

Problems marked $(\star)$ are bonus ones.

3.1. Consider a function $g : \mathrm{Mat}_2(\mathbb{C}) \times \mathrm{Mat}_2(\mathbb{C}) \to \mathbb{C}$ defined as

$$g(A, B) = 2\,\mathrm{tr}\,(AB) - \mathrm{tr}\,(A)\,\mathrm{tr}\,(B)$$

Show that

(a) g is an orthogonal inner product;

(b) g is degenerate, but its restriction on the subspace of matrices with zero trace is non-degenerate;

(c) restriction of g on the subspace of skew-Hermitian matrices is real and negative definite;

(d) find dim $\ker(g)$;

$(\star)$ do the same for $\mathrm{Mat}_n(\mathbb{C})$, $g(A, B) = n\,\mathrm{tr}\,(AB) - \mathrm{tr}\,(A)\,\mathrm{tr}\,(B)$.

3.2. Let L be non-degenerate orthogonal space. Show that for any two isotropic vectors $u, v \in L$ there exists an isometry of L taking u to v.

3.3. Show that any two maximal isotropic subspaces of a non-degenerate orthogonal space are isometric.

3.4. Let L be a complex linear space of dimension n with inner product g. Denote by r_0 dimension of $\ker g$. Show that dimension of any maximal isotropic subspace of L is equal to the integer part of $(n + r_0)/2$.

3.5. $(\star)$ Find maximal dimension of a subspace of $\mathrm{Mat}_2(\mathbb{R})$ consisting of degenerate matrices only.