Jacobs University
School of Engineering and Science

Spring Term 2011

ESM 2B, Homework 2

Due Date: 14:00 Wednesday, 23 February 2011.

Explain your answers! Problems marked $(\star)$ are bonus ones.

2.1. Which of the following maps from $\mathbb{R}^2$ to $\mathbb{R}^2$ are linear operators?

(a) $f(x, y) = (xy, x + y)$;

(b) $g(x, y) = (x + y, 2x)$.

2.2. Which of the following maps are linear transformations?

(a) $\mathcal{A} : \mathbb{R}^4 \to \mathbb{R}^3$, $\mathcal{A}(x, y, z, t) = (x - 2y, y - 2z, z + 2t)$;

$(\star)$ $\mathcal{A} : \mathbb{R}[x] \to \mathbb{R}[x]$, $(\mathcal{A}p)(x) = p(\alpha x^2 + \beta)$, where $\alpha, \beta \in \mathbb{R}$ are fixed numbers.

2.3. Find the kernel and the image of the following linear maps:

(a) $\mathcal{A} : \mathbb{R}^n \to \mathbb{R}^n$, $\mathcal{A}v = 0$;

(b) $\mathcal{A} : \mathbb{C}^n \to \mathbb{C}^n$, $\mathcal{A}v = v$;

(c) $\mathcal{A} : \mathbb{R}^3 \to \mathbb{R}^3$, $\mathcal{A}(x, y, z) = (y, z, 0)$;

(d) $\mathcal{A} : \mathbb{R}^4 \to \mathbb{R}^3$, $\mathcal{A}(x, y, z, t) = (x - y, y - z, z - t)$;

$(\star)$ $\mathcal{A} : \mathbb{R}[x] \to \mathbb{R}[x]$, $(\mathcal{A}p)(x) = p(\alpha x^2 + \beta)$, where $\alpha, \beta \in \mathbb{R}$ are fixed numbers.

2.4. Compute matrices of linear maps from Problem 2.3 in some bases. Which of the matrices do not depend on bases?

2.5. Let V be the space of polynomials with real coefficients of degree at most 2, and $\alpha \in \mathbb{R}$ is a fixed number, $\alpha \neq 0$. Define two maps $\mathcal{A}, \mathcal{B} : V \to V$ by

$$(\mathcal{A}p)(x) = p(\alpha x); \qquad (\mathcal{B}p)(x) = p(x - \alpha).$$

Show that $\mathcal{A}, \mathcal{B}$ are linear operators, and compute the matrices of $\mathcal{A}$ and $\mathcal{B}$ in the basis $\{1, x, x^2\}$.

2.6. Define $A = \begin{pmatrix} 2 & 0 & 1 \\ -1 & 3 & 1 \\ 0 & -2 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 2 & 0 \\ -1 & 2 \\ 2 & 3 \end{pmatrix}$, $C = \begin{pmatrix} 2 & 1 & 1 \end{pmatrix}$. Compute the following products if possible:

(a) AB; (b) BA; (c) CA; (d) C^2; (e) A^2; (f) AC; (g) CB; (h) CAB.

2.7. (a) Find all the matrices $X \in \mathrm{Mat}_2$ commuting with $A = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$.

$(\star)$ Find all the matrices $X \in \mathrm{Mat}_n$ commuting with $A = \begin{pmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \lambda_n \end{pmatrix}$.