Jacobs University
School of Engineering and Science

Spring Term 2011

ESM 2B, Homework 3

Due Date: 14:00 Wednesday, 2 March 2011.

Explain your answers! Problems marked ($\star$) are bonus ones.

3.1. Given matrices $A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 3 & 1 \\ 0 & 1 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 0 \\ 0 & 2 \\ 2 & 1 \end{pmatrix}$, $C = \begin{pmatrix} 2 & 0 & 1 \end{pmatrix}$, compute the following products if possible:

(a) CC^t; (b) C^tC; (c) C^tAB; (d) B^tA^tC; (e) BCA; (f) B^tC^tA.

3.2. Solve the following equation with respect $\{x, y, z\}$:

$$\left(\begin{pmatrix} x & y \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 4 & 0 \\ y & 2 \end{pmatrix}^t + \begin{pmatrix} 0 & -xy \\ 2 & -2 \end{pmatrix} \right)^t = \begin{pmatrix} 4 & 2z \\ 2 & -1 \end{pmatrix}$$

3.3. (a) Let $V = \mathbb{R}^3$, define an inner product on V by $\langle x \,|\, y \rangle = x_1y_3 + x_2y_2 + x_3y_1 + 2x_1y_1 + 2x_3y_3$. Write down the Gram matrix $G = (g_{ij})$ (where $g_{ij} = \langle e_i \,|\, e_j \rangle$) of the inner product in standard basis, and verify that $\langle x \,|\, y \rangle = x^tGy$.

($\star$) Let V be an inner product space, and G a Gram matrix in basis $\{e_i\}$, $i \leq n$. Let $\{f_j\}$, $j \leq n$ be another basis of V. For any $j \leq n$ one can write

$$f_j = \sum_{i=1}^{n} c_{ij} e_i$$

for some $c_{ij} \in \mathbb{F}$. Denote by C the matrix composed of the coefficients c_{ij}.

Show that the Gram matrix G' of the inner product in the basis $\{f_j\}$ is equal to $\overline{C^t}GC$.

3.4. Find all the solutions x of the following linear systems

$$(a) \begin{cases} x_1 + x_2 + 2x_3 = 1 \\ 2x_1 - x_2 = 3 \\ -2x_1 - x_3 = 2 \end{cases} \qquad (b) \begin{cases} x_1 - x_2 = 3 \\ + 2x_2 - x_3 = 10 \\ 3x_1 + 2x_2 + 5x_3 = -2 \end{cases}$$

$$(c) \begin{cases} 2x_1 - x_2 + x_3 = 3 \\ -x_1 + 2x_2 - x_3 = -5 \\ x_1 + x_2 = 2 \end{cases} \qquad (d) \begin{cases} 2x_1 - x_2 + x_3 = 5 \\ -x_1 + 2x_2 - x_3 = -4 \\ x_1 + x_2 = 1 \end{cases}$$

by identifying the coefficient matrix A, and then using Gauss elimination.

3.5. Compute the rank of the following matrices

$$(a) \begin{pmatrix} -1 & 1 & 3 \\ 1 & 6 & 2 \\ 4 & 2 & 0 \end{pmatrix} \quad (b) \begin{pmatrix} 0 & 1 & 1 \\ 3 & 2 & 3 \\ 2 & 0 & 1 \end{pmatrix} \quad (c) \begin{pmatrix} 1 & -1 & 1 & 0 \\ 3 & 2 & 1 & 1 \\ 4 & 0 & 2 & -1 \end{pmatrix} \quad (d) \begin{pmatrix} 2 & 0 & 3 \\ 2 & 1 & 1 \\ 3 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$$