Jacobs University
School of Engineering and Science

Spring Term 2011

ESM 2B, Homework 7

Due Date: 14:00 Wednesday, 6 April 2011.

Explain your answers! Problems marked ($\star$) are bonus ones.

7.1. Consider the vector space $C^1[0,1]$ of differentiable functions on $[0,1]$ with continuous derivative. Which of the following maps $\|\cdot\|\colon C^1[0,1] \to \mathbb{R}$ are norms?

(a) $\| f \| = \sup_{x\in[0,1]} |f(x)|$;

(b) $\| f \| = \sup_{x\in[0,1]} |f'(x)|$;

(c) $\| f \| = \sup_{x\in[0,1]} |f(x) - f(0)|$;

(d) $\| f \| = \sup_{x\in[0,1]} |f^5(x)|$;

(e) $\| f \| = \sup_{x\in[0,1]} |f(x) + f'(x)|$;

(f) $\| f \| = \sup_{x\in[0,1]} |f(0) + f'(x) + f(1)|$.

7.2. Let P_3 be the vector space of polynomials with real coeffcients of degree less than or equal to 3. For any $p, q \in P_3$ consider the inner product

$$\langle p|q\rangle = \int_0^1 p(x)q(x)\,dx$$

Apply the Gram-Schmidt procedure to the basis $\{1, x, x^2, x^3\}$ to obtain an orthonormal basis.

7.3. Consider the vector space of continuous real-valued functions on $[0, 2\pi]$ with inner product

$$\langle f|g\rangle = \int_0^{2\pi} f(x)g(x)\,dx$$

Show that any two distinct functions from the set $\{1, \cos nx, \sin nx \mid n\in\mathbb{N}\}$ are mutually orthogonal.

7.4. (a) Show that any norm induced by inner product satisfies the *parallelogramm identity*

$$\| x + y \|^2 + \| x - y \|^2 = 2\| x \|^2 + 2\| y \|^2$$

(b) Give an example of two continuous functions f, g on $[0, 1]$ for which

$$\| f + g \|_\infty^2 + \| f - g \|_\infty^2 \neq 2\| f \|_\infty^2 + 2\| g \|_\infty^2$$

where $\| f \|_\infty$ is the norm defined in Problem 1(a).

7.5. ($\star$) Define the vector space ℓ^p as the space of all infinite sequences $\{x_n\}$ of real numbers such that the series $\sum_{n=1}^{\infty} |x_n|^p$ converges. For which p and k the inclusion $\ell^p \subset \ell^{p+k}$ holds?