Jacobs University
School of Engineering and Science Spring Term 2011

Linear Algebra II, Homework 12

Due Date: Wednesday, May 11, in class.

Problems marked ($\star$) are bonus ones.

12.1. Let $f \in \mathrm{End}(L)$, $g \in \mathrm{End}(M)$, and dimensions of L and M are n and m respectively. Define a linear operator $f \otimes g \in \mathrm{End}(L \otimes M)$ on factorizable tensors by

$$(f \otimes g)(l \otimes m) = f(l) \otimes g(m)$$

for any $l \in L, m \in M$. Show that

(a) $f \otimes g$ is well defined; (b) $\mathrm{tr}\ f \otimes g = \mathrm{tr}\ f\ \mathrm{tr}\ g$; (c)($\star$) $\det f \otimes g = (\det f)^m (\det g)^n$.

12.2. Let $\{e_1, \ldots, e_n\}$ be a basis of L, and $\{e^1, \ldots, e^n\}$ be a dual basis of L^*. Compute the contraction of the tensor

(a) $T = e^j \otimes e_i$; (b) $T = \sum_{j=1}^{n} e^j \otimes e_{n-j}$; (c) $T = \sum_{i,j=1}^{n} (-1)^{i+j} e^j \otimes e_i$.

12.3. (a) Show that factorizable $(1,1)$-tensors are exactly linear operators of rank one.

(b) Let $f = \alpha \otimes x$, $g = \beta \otimes y$ be linear operators, where $\alpha, \beta \in L^*$, $x, y \in L$. Compute the contraction

$$f \otimes g \to \alpha(y) \beta \otimes x$$

in terms of f and g.

12.4. Let $f_1 : L_1 \to M_1$, $f_2 : L_2 \to M_2$ be linear maps. Show that

(a) $\mathrm{im}\,(f_1 \otimes f_2) = \mathrm{im}\,(f_1) \otimes \mathrm{im}\,(f_2)$;

(b) $\mathrm{im}\,(f_1 \otimes f_2) = (\mathrm{im}\,(f_1) \otimes M_2) \cap (M_1 \otimes \mathrm{im}\,(f_2))$;

(c) $\ker(f_1 \otimes f_2) = (\mathrm{im}\,(f_1) \otimes M_2) + (M_1 \otimes \mathrm{im}\,(f_2))$.