Jacobs University
School of Engineering and Science

Spring Term 2011

Linear Algebra II, Homework 13

Due Date: Tuesday, May 17.

Problems marked ($\star$) are bonus ones.

13.1. Let dim $L = 4$, $f = e^1 \otimes e_2 + e^2 \otimes e_3 + e^3 \otimes e_4 \in T_1^1(L)$. Find all

(a) $\alpha \in L^*$, such that $f(x, \alpha) = 0$ for every $x \in L$;

(b) $x \in L$, such that $f(x, \alpha) = 0$ for every $\alpha \in L^*$.

13.2. (a) Show that for every $f \in \Lambda^2(L)$ there is a basis $(e_1, \ldots, e_n)$ of L such that

$$f = e_1 \wedge e_2 + e_3 \wedge e_4 + \cdots + e_{k-1} \wedge e_k$$

for some even k.

(b) Show that $f \in \Lambda^2(L)$ is factorizable if and only if $f \wedge f = 0$.

13.3. Let $\{e_1, \ldots, e_n\}$ be a basis of L. Show that symmetric tensors of the form

$$e_1^{a_1} \ldots e_n^{a_n} \ (= S(\underbrace{e_1 \otimes \cdots \otimes e_1}_{a_1} \otimes \underbrace{e_2 \otimes \cdots \otimes e_2}_{a_2} \otimes \cdots \otimes \underbrace{e_n \otimes \cdots \otimes e_n}_{a_n})), \qquad a_1 + \cdots + a_n = q$$

span the symmetric power $S^q(L)$.

13.4. ($\star$) Let $f \in \mathrm{End}(L)$, $\{e_1, \ldots, e_n\}$ is a basis of L. Define a *symmetric square* $S^2(f) \in \mathrm{End}(S^2(L))$ by

$$(S^2(f))(e_i e_j) = S(f(e_i) \otimes f(e_j))$$

(a) Show that $S^2(f)$ is well defined.

(b) Show that $\mathrm{tr}\ S^2(f) = \frac{1}{2}((\mathrm{tr}\ f)^2 + \mathrm{tr}\ f^2)$.

(c) Suppose that the characteristic polynomial of f has roots $\lambda_1, \ldots, \lambda_n$. Show that the characteristic polynomial of $S^2(f)$ has $n(n+1)/2$ roots $\lambda_i \lambda_j$, $1 \leq i \leq j \leq n$.