Linear Algebra II, Homework 3

Due Date: Wednesday, March 2, in class.

Problems marked ($\star$) are bonus ones.

3.1. Let L be a linear space, $M \subset L$ is a subspace. The *annihilator* $\operatorname{Ann} M$ is a subset of L^* defined as follows:

$$\operatorname{Ann} M = \{f \in L^* | f(l) = 0 \ \forall l \in M\}$$

(a) Show that $\operatorname{Ann} M$ is a linear space.

(b) Assuming L to be finite-dimensional, construct explicitely an isomorphism

$$L^*/\operatorname{Ann} M \to M^*$$

(c) Show that $\quad \dim M + \dim \operatorname{Ann} M = \dim L.$

3.2. Let L be an 3-dimensional real linear space with inner product g. Find the signature of g if the Gram matrix G of g in some basis looks like

(a) $G = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$; (b) $G = \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$.

3.3. Let (L, g) be a real $(n+1)$-dimensional linear space of signature $(n, 1)$. Let $L_0 \subset L$ be 1-dimensional subspace, $v \in L$, $v \neq 0$. Show that

(a) if $(v, v) > 0$, then $L_0^\perp$ has signature $(n-1, 1)$;

(b) if $(v, v) < 0$, then $L_0^\perp$ is positive definite;

(c) if v is isotropic, then $L_0^\perp$ is degenerate, and its signature is $(n-1, 0, 1)$.

3.4. Show that if (L, g) is an inner product space, and $v \in L$ is not isotropic, then the map

$$x \to x - \frac{2(x, v)}{(v, v)} v$$

(called *reflection in* v) is an isometry of L.

3.5. ($\star$) Let M_2 be the space of all symmetric 2×2 real matrices.

(a) Show that the formula

$$(A, B) = \frac{1}{2}(\det(A + B) - \det A - \det B)$$

defines an inner product on M_2.

(b) Find the signature of the inner product defined in (a).