Jacobs University
School of Engineering and Science Spring Term 2011

Linear Algebra II, Homework 7

Due Date: Wednesday, March 30, in class.

Problems marked ($\star$) are bonus ones.

7.1. Let L be a unitary space. For every self-adjoint operator f define an inner product

$$g_f(x, y) = (f(x), y)$$

Show that g_f is a well-defined Hermitian inner product, and the map $f \to g_f$ is a bijection between the set of self-adjoint operators and Hermitian forms.

7.2. Give an example of two quadratic forms that cannot be simultaneously diagonalized.

7.3. Let A be a real square matrix. Show that A is symmetric if and only if there exist real invertible matrix C and real diagonal matrix D with

$$A = C^{-1}DC$$

7.4. Let f be a normal operator in a unitary space L. Show that

(a) kernels of f and f^* coincide;

(b) $\operatorname{im} f = \ker f^*$;

(c) images of f and f^* coincide;

(d) L is a direct orthogonal sum of the kernel and image of f.

A self-adjoint operator f is *non-negative* if $(f(x), x) \geq 0$ for every x, and *positive* if $(f(x), x) > 0$ for every $x \neq 0$.

7.5. (a) Given any operator f on a unitary space, show that f^*f is a non-negative self-adjoint operator, and it is positive if and only if f is invertible.

(b) Show that for any non-negative self-adjoint operator f there exists non-negative self-adjoint operator h such that $f = h^2$.

7.6. ($\star$) Let f_1, f_2 be positive self-adjoint operators. Show that

(a) if h is self-adjoint, $h^2 = f_1$, and f_1 and f_2 commute, then h and f_2 commute;

(b) $f_1 f_2$ is positive self-adjoint if and only if f_1 and f_2 commute.