Jacobs University
School of Engineering and Science

Spring Term 2011

Linear Algebra II, Homework 8

Due Date: Wednesday, April 6, in class.

Problems marked ($\star$) are bonus ones.

8.1. Show that an operator f in a unitary space is normal if and only if every eigenvector of f is an eigenvector of f^*.

8.2. Let L be an orthogonal space with form

$$(x, y) = -x_0 y_0 + \sum_{i=1}^{n} x_i y_i$$

Let $(x, x) < 0$ and $(y, y) < 0$. Show that $(x, y) < 0$ if and only if $x_0 y_0 > 0$.

8.3. Let $\mathcal{M}$ be Minkowski space. Show that

(a) for every two non-collinear time-like vectors $x, y \in \mathcal{M}$ there exists $f \in SO^+(3,1)$ taking x to y;

(b) for every two non-intersecting 2-planes Π_1 and Π_2 intersecting the light cone there exists $f \in SO^+(3,1)$ taking Π_1 to Π_2;

(c) there is a basis of $\mathcal{M}$ consisting of isotropic vectors.

8.4. ($\star$) **(Polar decomposition)**

Let f be an invertible operator in a unitary space. Define r_1, r_2 to be positive self-adjoint operators, such that $r_1^2 = f f^*$, $r_2^2 = f^* f$ (see Problem 7.5b).

(a) Show that there exist unitary operators u_1, u_2, such that

$$f = r_1 u_1 = u_2 r_2$$

The representations above are called *polar decompositions* of f.

(b) Show that polar decompositions $f = r_1 u_1 = u_2 r_2$ are unique.