Durham University
Pavel Tumarkin

Epiphany 2012

Riemannian Geometry IV, Homework 1 (Week 12)

Due date for starred problems: **Wednesday, February 15**.

1.1. Rescaling Lemma
Let $c : [0, a] \to M$ be a geodesic, and $k > 0$. Define a curve γ by

$$\gamma : [0, a/k] \to M, \qquad \gamma(t) = c(kt)$$

Show that γ is geodesic with $\gamma'(t) = kc'(kt)$.

1.2. ($\star$) **First Variation Formula of energy**
Let $F : (-\epsilon, \epsilon) \times [a, b] \to M$ be a variation of a differentiable curve $c : [a, b] \to M$ with $c'(t) \neq 0$ for all $t \in [a, b]$ and X be its variational vector field. Let $E : (-\epsilon, \epsilon)$ denote the associated energy, i.e.,

$$E(s) = \frac{1}{2} \int_a^b \| \frac{\partial F}{\partial t}(s, t) \|^2 dt.$$

(a) Show that

$$E'(0) = \langle X(b), c'(b) \rangle - \langle X(a), c'(a) \rangle - \int_a^b \langle X(t), \frac{D}{dt} c'(t) \rangle dt.$$

Simplify the formula for the cases when

(b) c is a geodesic,

(c) F is a proper variation,

(d) c is a geodesic and F is a proper variation.

Let $c : [a, b] \to M$ be a curve connecting p and q (not necessarily parametrized proportional to arc length). Show that

(e) $E'(0) = 0$ for every proper variation implies that c is a geodesic.

(f) Assume that c minimizes the energy amongst all curves $\gamma : [a, b] \to M$ which connect p and q. Then c is a geodesic.