Durham University
Pavel Tumarkin Epiphany 2012

Riemannian Geometry IV, Homework 2 (Week 13)

Due date for starred problems: **Wednesday, February 15**.

2.1. Let (M, g) be a Riemannian manifold and $p \in M$. Let $\epsilon > 0$ be small enough such that

$$\exp_p : B_\epsilon(0_p) \to B_\epsilon(p) \subset M$$

is a diffeomorphism. Let $\gamma : [0,1] \to B_\epsilon(p) \setminus \{p\}$ be any curve.

Show that there exist a curve $v : [0,1] \to T_pM$, $\|v(s)\| = 1$ for all $s \in [0,1]$, and a non-negative function $r : [0,1] \to \mathbb{R}_{\geq 0}$, such that

$$\gamma(s) = \exp_p(r(s)v(s))$$

2.2. ($\star$) **Geodesic normal coordinates**
Let (M, g) be a Riemannian manifold and $p \in M$. Let $\epsilon > 0$ such that

$$\exp_p : B_\epsilon(0_p) \to B_\epsilon(p) \subset M$$

is a diffeomorphism. Let $v_1, \ldots, v_n$ be an orthonormal basis of T_pM. Then a local coordinate chart of M is given by $\varphi = (x_1, \ldots, x_n) : B_\epsilon(p) \to V := \{w \in \mathbb{R}^n \mid \|w\| < \epsilon\}$ via

$$\varphi^{-1}(x_1, \ldots, x_n) = \exp_p(\sum_{i=1}^{n} x_i v_i).$$

The coordinate functions $x_1, \ldots, x_n$ of φ are called *geodesic normal coordinates*.

(a) Let g_{ij} be the first fundamental form in terms of the above coordinate system φ. Show that at $p \in M$:

$$g_{ij}(p) = \delta_{ij} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

(b) Let $w = (w_1, \ldots, w_n) \in \mathbb{R}^n$ be arbitrary vector, and $c(t) = \varphi^{-1}(tw)$. Explain why $c(t)$ is a geodesic and deduce from this fact that

$$\sum_{i,j} w_i w_j \Gamma_{ij}^k(c(t)) = 0,$$

for all $1 \leq k \leq n$.

(c) Derive from (b) that all Christoffel symbols Γ_{ij}^k of the chart φ vanish at the point $p \in M$.

2.3. Let (M, g) be a Riemannian manifold and $v_1, \ldots, v_n \in T_pM$ be an orthonormal basis. As it follows from problem 2.2, for the geodesic normal coordinates $\varphi : B_\epsilon(p) \to B_\epsilon(0) \subset \mathbb{R}^n$,

$$\varphi^{-1}(x_1, \ldots, x_n) = \exp_p(\sum x_i v_i)$$

we have $\frac{\partial}{\partial x_i}|_p = v_i$ and $\nabla_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_j} = 0$.

Define an *orthonormal frame* $E_1, \ldots, E_n : B_\epsilon(p) \to TM$ (i.e. an n-tuple of vector fields composing an orthonormal basis of T_qM in every point $q \in B_\epsilon(p)$) by Gram-Schmidt orthonormalisation, i.e.,

$$\begin{aligned} F_1(q) &:= \left.\frac{\partial}{\partial x_1}\right|_q, \qquad E_1(q) := \frac{1}{\|F_1(q)\|} F_1(q), \\ &\cdots \\ F_k(q) &:= \left.\frac{\partial}{\partial x_k}\right|_q - \sum_{j=1}^{k-1} \left\langle \left.\frac{\partial}{\partial x_k}\right|_q, E_j(q) \right\rangle E_j(q), \qquad E_k(q) := \frac{1}{\|F_k(q)\|} F_k(q), \\ &\cdots \end{aligned}$$

As you might have shown in problem 2.2, $E_i(p) = v_i$ and $E_1(q), \ldots, E_n(q)$ are orthonormal in T_qM for all $q \in B_\epsilon(p)$. Show that

$$(\nabla_{E_i} E_j)(p) = 0$$

for all $i, j \in \{1, \ldots, n\}$.

Hint: Prove first by induction over k that

$$\begin{aligned} \left(\nabla_{\frac{\partial}{\partial x_i}} F_k\right)(p) &= 0, \\ \nabla_{\frac{\partial}{\partial x_i}} \langle F_k, F_k \rangle^{-1/2}(p) &= 0, \\ \left(\nabla_{\frac{\partial}{\partial x_i}} E_k\right)(p) &= 0, \end{aligned}$$

for all $i \in \{1, \ldots, n\}$.