Durham University
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Epiphany 2012

Riemannian Geometry IV, Homework 3 (Week 14)

Due date for starred problems: **Wednesday, February 15**.

3.1. ($\star$) Consider the upper half-plane $M = \{(x, y) \in \mathbb{R}^2 \mid y > 0\}$ with the metric

$$(g_{ij}) = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{y} \end{pmatrix}$$

(a) Show that all the Christoffel symbols are zero except $\Gamma^2_{22} = -\frac{1}{2y}$.

(b) Show that the vertical segment $x = 0$, $\varepsilon \leq y \leq 1$ with $0 < \varepsilon < 1$ is a geodesic curve when parametrized proportionally to arc length.

(c) Show that the length of the segment $x = 0$, $\varepsilon \leq y \leq 1$ with $0 < \varepsilon < 1$ tends to 2 as ε tends to zero.

(d) Show that (M, g) is not geodesically complete.

3.2. Let (M, g) be a Riemannian manifold and R its curvature tensor. Let $f, g, h \in C^\infty(M)$, and X, Y, Z, W be vector fields on M. Show that

(a) $R(fX, Y)Z = fR(X, Y)Z$;

(b) $R(X, fY)Z = fR(X, Y)Z$;

(c) $\langle R(X, Y)fZ, W\rangle = \langle fR(X, Y)Z, W\rangle$;

(d) $R(fX, gY)hZ = fghR(X, Y)Z$.

3.3. Let (M, g) be a Riemannian manifold and R its curvature tensor. Prove the *First Bianchi Identity*:

$$R(X, Y)Z + R(Y, Z)X + R(Z, X)Y = 0$$

for X, Y, Z vector fields on M by reducing the equation to Jacobi identity.