Durham University
Pavel Tumarkin

Epiphany 2012

Riemannian Geometry IV, Homework 5 (Week 16)

Due date for starred problems: **Wednesday, March 14**.

5.1. Second Variation Formula of Energy
Let $F : (-\epsilon, \epsilon) \times [a, b] \to M$ be a proper variation of a geodesic $c : [a, b] \to M$, and let X be its variational vector field. Let $E : (-\epsilon, \epsilon) \to \mathbb{R}$ denote the associated energy, i.e.,

$$E(s) = \frac{1}{2} \int_a^b \| \frac{\partial F}{\partial t}(s,t) \|^2 dt.$$

Show that

$$E''(0) = \int_a^b \| \frac{D}{dt} X \|^2 - \langle R(X, c')c', X \rangle dt$$

5.2. ($\star$) A Riemannian manifold (M, g) is called *Einstein manifold* if there exists $c \in \mathbb{R}$ such that

$$Ric_p(v, w) = c\langle v, w \rangle$$

for every $p \in M$, $v, w \in T_pM$.

(a) Show that (M, g) is Einstein manifold if and only if there exists $c \in \mathbb{R}$ such that

$$Ric(v) = c$$

for every unit tangent vector v.

(b) Show that if (M, g) is of constant sectional curvature then (M, g) is Einstein manifold.

5.3. Let (M, g) be a Riemannian manifold, $\mathcal{X}(M)$ be the vector space of smooth vector fields on M, and ∇ be the Levi-Civita connection. Recall that a map

$$A : \mathcal{X}(M) \times \cdots \times \mathcal{X}(M) \to C^\infty(M) \text{ or } \mathcal{X}(M)$$

is a *tensor* if it is linear in each argument, i.e.,

$$A(X_1, \cdots, fX_i + gY_i, \cdots, X_r) = fA(X_1, \cdots, X_i, \cdots, X_r) + gA(X_1, \cdots, Y_i, \cdots, X_r),$$

for all $X, Y \in \mathcal{X}(M)$ and $f, g \in C^\infty(M)$.

(a) Let

$$T : \underbrace{\mathcal{X}(M) \times \cdots \times \mathcal{X}(M)}_{r \text{ factors}} \to C^\infty(M)$$

be a tensor. The *covariant derivative* of T is a map

$$\nabla T : \underbrace{\mathcal{X}(M) \times \cdots \times \mathcal{X}(M)}_{r+1 \text{ factors}} \to C^\infty(M),$$

defined by

$$\nabla T(X_1, \dots, X_r, Y) = Y(T(X_1, \dots, X_r)) - \textstyle\sum_{j=1}^{r} T(X_1, \dots, \nabla_Y X_j, \dots, X_r).$$

Show that ∇T is a tensor.

Tensor T is called *parallel* if $\nabla T = 0$.

(b) Assume that $T_1, T_2 : \mathcal{X} \times \mathcal{X} \to C^\infty(M)$ are parallel tensors. Show that the tensor $T : \mathcal{X} \times \mathcal{X} \times \mathcal{X} \times \mathcal{X} \to C^\infty(M)$, defined as

$$T(X_1, X_2, X_3, X_4) = T_1(X_1, X_2) T_2(X_3, X_4),$$

is also parallel.

(c) Use (b) to show that $\nabla R' = 0$ for the tensor

$$R'(X, Y, Z, W) = \langle X, W \rangle \langle Y, Z \rangle - \langle X, Z \rangle \langle Y, W \rangle.$$

(d) Use (c) and Problem 4.2 to show that all manifolds with constant sectional curvature have parallel Riemann curvature tensor

$$R(X, Y, Z, W) := \langle R(X, Y)Z, W \rangle.$$