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Epiphany 2013

Riemannian Geometry IV, Homework 1 (Week 12)

Due date for starred problems: **Tuesday, February 12**.

1.1. Lie bracket of vector fields

Let $X, Y \in \Gamma(TM)$, $X = \sum_{i=1}^{n} a_i \frac{\partial}{\partial x_i}$, $Y = \sum_{j=1}^{n} b_j \frac{\partial}{\partial x_j}$. Compute the Lie bracket

$$[X, Y] = XY - YX$$

in coordinates and show that it is a vector field on M.

Solution: Applying $[X, Y]$ to a function f, we obtain

$$[X,Y]f = XYf - YXf = X\left(\sum_{j=1}^{n} b_j \frac{\partial f}{\partial x_j}\right) - Y\left(\sum_{i=1}^{n} a_i \frac{\partial f}{\partial x_i}\right) =$$

$$= \sum_{i=1}^{n} a_i \sum_{j=1}^{n} \frac{\partial}{\partial x_i}\left(b_j \frac{\partial f}{\partial x_j}\right) - \sum_{j=1}^{n} b_j \sum_{i=1}^{n} \frac{\partial}{\partial x_j}\left(a_i \frac{\partial f}{\partial x_i}\right) =$$

$$= \sum_{i=1}^{n} a_i \sum_{j=1}^{n} \left(\frac{\partial b_j}{\partial x_i}\frac{\partial f}{\partial x_j} + b_j \frac{\partial^2 f}{\partial x_i \partial x_j}\right) - \sum_{j=1}^{n} b_j \sum_{i=1}^{n} \left(\frac{\partial a_i}{\partial x_j}\frac{\partial f}{\partial x_i} + a_i \frac{\partial^2 f}{\partial x_j \partial x_i}\right) =$$

$$= \sum_{i,j=1}^{n} \left(a_i \frac{\partial b_j}{\partial x_i}\frac{\partial f}{\partial x_j} + a_i b_j \frac{\partial^2 f}{\partial x_j \partial x_i}\right) - \sum_{i,j=1}^{n} \left(b_j \frac{\partial a_i}{\partial x_j}\frac{\partial f}{\partial x_i} + b_j a_i \frac{\partial^2 f}{\partial x_i \partial x_j}\right) =$$

$$= \sum_{i,j=1}^{n} a_i \frac{\partial b_j}{\partial x_i}\frac{\partial f}{\partial x_j} - \sum_{i,j=1}^{n} b_j \frac{\partial a_i}{\partial x_j}\frac{\partial f}{\partial x_i} = \sum_{i=1}^{n}\left(\sum_{j=1}^{n}\left(a_j \frac{\partial b_i}{\partial x_j} - b_j \frac{\partial a_i}{\partial x_j}\right)\right)\frac{\partial f}{\partial x_i},$$

which shows, in particular, that $[X, Y]$ is a vector field since it is a linear combination of $\left\{\frac{\partial}{\partial x_i}\right\}$.

1.2. The equation of geodesic

Let $c(t)$ be a curve on M, and $X \in \Gamma(c^{-1}TM)$, $X(t) = \sum_{i=1}^{n} a_i(t) \frac{\partial}{\partial x_i}$. Recall that covariant derivative ∇_t of X along $c(t)$ is given by

$$\nabla_t X = \sum_{i=1}^{n}\left(a_i'(t)\frac{\partial}{\partial x_i} + a_i(t)\nabla_{c'(t)}\frac{\partial}{\partial x_i}\right)$$

Use the formula above and the definitions of connection and Christoffel symbols to show that $c(t)$ is geodesic (i.e., $\nabla_t c'(t) = 0$) if and only if for any $k = 1, \ldots, n$

$$c_k''(t) + \sum_{i,j=1}^{n} c_i'(t) c_j'(t) \Gamma_{ij}^k = 0$$

Solution: By the definition of covariant derivative,

$$\nabla_t c'(t) = \sum_i c_i''(t)\frac{\partial}{\partial x_i} + \sum_i c_i'(t)\nabla_{c'(t)}\frac{\partial}{\partial x_i}$$

Therefore, if $c(t)$ is a geodesic, we have

$$0 = \nabla_t c'(t) = \sum_i c_i''(t)\frac{\partial}{\partial x_i} + \sum_i c_i'(t)\nabla_{\sum_j c_j'(t)\frac{\partial}{\partial x_j}}\frac{\partial}{\partial x_i} =$$

$$= \sum_i c_i''(t)\frac{\partial}{\partial x_i} + \sum_i c_i'(t)\sum_j c_j'(t)\nabla_{\frac{\partial}{\partial x_j}}\frac{\partial}{\partial x_i} = \sum_i c_i''(t)\frac{\partial}{\partial x_i} + \sum_i c_i'(t)\sum_j c_j'(t)\sum_k \Gamma_{ij}^k\frac{\partial}{\partial x_k} =$$

$$= \sum_k c_k''(t)\frac{\partial}{\partial x_k} + \sum_k\left(\sum_{i,j} c_i'(t)c_j'(t)\Gamma_{ij}^k\right)\frac{\partial}{\partial x_k} = \sum_k\left(c_k''(t) + \sum_{i,j} c_i'(t)c_j'(t)\Gamma_{ij}^k\right)\frac{\partial}{\partial x_k}$$

The equality shows that every component of the vector above is zero.

1.3. ($\star$) **Rescaling Lemma**
Let $c : [0, a] \to M$ be a geodesic, and $k > 0$. Define a curve γ by

$$\gamma : [0, a/k] \to M, \qquad \gamma(t) = c(kt)$$

Show that γ is geodesic with $\gamma'(t) = kc'(kt)$.

Solution: Proof is sraightforward: all the entries of the corresponding differential equation for $c(t)$ are multiplied by k^2.

1.4. Let (M, g) be a Riemannian manifold and $p \in M$. Let $\epsilon > 0$ be small enough such that

$$\exp_p : B_\epsilon(0_p) \to B_\epsilon(p) \subset M$$

is a diffeomorphism. Let $\gamma : [0, 1] \to B_\epsilon(p) \setminus \{p\}$ be any curve.

Show that there exist a curve $v : [0, 1] \to M_p$, $\|v(s)\| = 1$ for all $s \in [0, 1]$, and a non-negative function $r : [0, 1] \to \mathbb{R}_{\geq 0}$, such that

$$\gamma(s) = \exp_p(r(s)v(s))$$

Solution: Since $\exp_p : B_\epsilon(0_p) \to B_\epsilon(p)$ is a diffeomorphism, for every $s \in [0, 1]$ the point $\gamma(s)$ can be represented as $\exp_p(w(s))$ for some $w(s) \in B_\epsilon(0_p)$. Define $r(s) = \|w(s)\|$, $v(s) = w(s)/r(s)$.

1.5. Let (M, g) be a Riemannian manifold and R its curvature tensor. Let $f, g, h \in C^\infty(M)$, and X, Y, Z, W be vector fields on M. Show that

(a) $R(fX, Y)Z = fR(X, Y)Z$;

(b) $R(X, fY)Z = fR(X, Y)Z$;

(c) $\langle R(X, Y)fZ, W\rangle = \langle fR(X, Y)Z, W\rangle$;

(d) $R(fX, gY)hZ = fghR(X, Y)Z$.

Solution:

(a) Note that $[fX, Y] = f[X, Y] - (Yf)X$. We have

$$\begin{aligned} R(fX, Y)Z &= -(\nabla_{fX}\nabla_Y Z - \nabla_Y \nabla_{fX} Z) + \nabla_{[fX,Y]} Z = \\ &= -f\nabla_X\nabla_Y Z + \nabla_Y(f\nabla_X Z) + \nabla_{f[X,Y]-(Yf)X} Z = \\ &= -f\nabla_X\nabla_Y Z + (Yf)\nabla_X Z + f\nabla_Y\nabla_X Z + f\nabla_{[X,Y]} Z - (Yf)\nabla_X Z = \\ &= -f(\nabla_X\nabla_Y Z + \nabla_Y\nabla_X Z + \nabla_{[X,Y]} Z) = fR(X, Y)Z. \end{aligned}$$

(b) Using the symmetry $R(X, Y)Z = -R(Y, X)Z$, we conclude with (a) that

$$R(X, fY)Z = -R(fY, X)Z = -fR(Y, X)Z = fR(X, Y)Z.$$

(c) Using the symmetry $\langle R(X, Y)Z, W\rangle = \langle R(Z, W)X, Y\rangle$ twice, we conclude with (a) that

$$\begin{aligned} \langle R(X, Y)fZ, W\rangle &= \langle R(fZ, W)X, Y\rangle = \langle fR(Z, W)X, Y\rangle = \\ &= f\langle R(Z, W)X, Y\rangle = f\langle R(X, Y)Z, W\rangle = \langle fR(X, Y)Z, W\rangle. \end{aligned}$$

(d) Since (c) holds for all vector fields W, we conclude that

$$R(X, Y)fZ = fR(X, Y)Z.$$

Using this together with (a) and (b), we obtain

$$R(fX, gY)hZ = fghR(X, Y)Z.$$

1.6. First Bianchi Identity

Let (M, g) be a Riemannian manifold and R its curvature tensor. Prove the *First Bianchi Identity*:

$$R(X, Y)Z + R(Y, Z)X + R(Z, X)Y = 0$$

for X, Y, Z vector fields on M by reducing the equation to *Jacobi identity*

$$[X, [Y, Z]] + [Z, [X, Y]] + [Y, [Z, X]] = 0$$

Solution: We have

$$\begin{aligned} &-(R(X, Y)Z + R(Y, Z)X + R(Z, X)Y) = (\nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]} Z) + \\ &+ (\nabla_Y\nabla_Z X - \nabla_Z\nabla_Y X - \nabla_{[Y,Z]} X) + (\nabla_Z\nabla_X Y - \nabla_X\nabla_Z Y - \nabla_{[Z,X]} Y) = \\ &= \nabla_X(\nabla_Y Z - \nabla_Z Y) + \nabla_Y(\nabla_Z X - \nabla_X Z) + \nabla_Z(\nabla_X Y - \nabla_Y X) - (\nabla_{[X,Y]} Z) + \nabla_{[Y,Z]} X) + \nabla_{[Z,X]} Y) = \\ &= \nabla_X[Y, Z] + \nabla_Y[Z, X] + \nabla_Z[X, Y] - (\nabla_{[X,Y]} Z) + \nabla_{[Y,Z]} X) + \nabla_{[Z,X]} Y) = \\ &= (\nabla_X[Y, Z] - \nabla_{[Y,Z]} X) + (\nabla_Y[Z, X] - \nabla_{[Z,X]} Y) + (\nabla_Z[X, Y] - \nabla_{[X,Y]} Z) = \\ &= -([[Y, Z], X] + [[Z, X], Y] + [[X, Y], Z]) = 0. \end{aligned}$$

1.7. Constant sectional curvature of real hyperbolic n-space

Let $\mathbb{H}^n$ be the upper halfspace model of the real hyperbolic n-space

$$\mathbb{H}^n = \{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n \mid x_n > 0\}$$

Recall that the hyperbolic metric $\tilde{g}$ on $\mathbb{H}^n$ is given by

$$\tilde{g}_{ii} = \frac{1}{x_n^2}, \qquad \tilde{g}_{ij} = 0 \qquad \text{for} \quad i \neq j$$

Consider first $n = 3$.

(a) Compute the Christoffel symbols of $(\mathbb{H}^3, \tilde{g})$.

(b) Show that sectional curvatures $K(\frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3})$, $K(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_3})$, $K(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2})$ all are equal to -1 at every point of $\mathbb{H}^3$.

(c) Use (b) and the linearity of Riemann curvature tensor to show that real hyperbolic 3-space has constant sectional curvature.

Consider now the general case.

(d) Compute the Christoffel symbols of $(\mathbb{H}^n, \tilde{g})$. (The computations are very similar to (a)).

(e) Show that for every $a = (a_1, a_2, \ldots, a_n) \in \mathbb{H}^n$ and every $i, j \in [1, \ldots, n-1]$ the submanifold

$$N = \{x \in \mathbb{H}^n \mid x_k = a_k \quad \text{for all} \quad k \neq i, j, n\}$$

with the metric induced from $\mathbb{H}^n$ is a real hyperbolic 3-space.

(f) Show that $K(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}) = -1$ for all pairs (i, j) at every point of $\mathbb{H}^n$.

(g) Use (f) and the linearity of Riemann curvature tensor to show that real hyperbolic n-space has constant sectional curvature.

Solution:

(a) We use the formula

$$\Gamma_{ij}^k = \frac{1}{2} \sum_{r=1}^{n} g^{kr}(g_{ir,j} + g_{jr,i} - g_{ij,r})$$

The only non-zero $g_{ij,k}$ are $g_{ii,3} = -2/x_3^3$. Thus, the only non-zero Christoffel symbols are

$$\Gamma_{11}^3 = \Gamma_{22}^3 = \frac{1}{x_3}, \quad \Gamma_{33}^3 = \Gamma_{13}^1 = \Gamma_{31}^1 = \Gamma_{23}^2 = \Gamma_{32}^2 = -\frac{1}{x_3},$$

the remaining ones are zero. Using this, we compute that

$$\nabla_{\frac{\partial}{\partial x_1}} \frac{\partial}{\partial x_1} = \nabla_{\frac{\partial}{\partial x_2}} \frac{\partial}{\partial x_2} = \Gamma_{11}^3 \frac{\partial}{\partial x_3} = \Gamma_{22}^3 \frac{\partial}{\partial x_3} = \frac{1}{x_3} \frac{\partial}{\partial x_3}, \quad \nabla_{\frac{\partial}{\partial x_3}} \frac{\partial}{\partial x_3} = \Gamma_{33}^3 \frac{\partial}{\partial x_3} = -\frac{1}{x_3} \frac{\partial}{\partial x_3},$$
$$\nabla_{\frac{\partial}{\partial x_1}} \frac{\partial}{\partial x_2} = \nabla_{\frac{\partial}{\partial x_2}} \frac{\partial}{\partial x_1} = 0, \quad \nabla_{\frac{\partial}{\partial x_1}} \frac{\partial}{\partial x_3} = \nabla_{\frac{\partial}{\partial x_3}} \frac{\partial}{\partial x_1} = -\frac{1}{x_3} \frac{\partial}{\partial x_1}, \quad \nabla_{\frac{\partial}{\partial x_2}} \frac{\partial}{\partial x_3} = \nabla_{\frac{\partial}{\partial x_3}} \frac{\partial}{\partial x_2} = -\frac{1}{x_3} \frac{\partial}{\partial x_2}.$$

(b) First, we compute $K(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_3})$.

$$K(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_3}) = \frac{\left\langle R(\frac{\partial}{\partial x_3}, \frac{\partial}{\partial x_1})\frac{\partial}{\partial x_3}, \frac{\partial}{\partial x_1} \right\rangle}{\|\frac{\partial}{\partial x_1}\|^2 \|\frac{\partial}{\partial x_3}\|^2 - \left\langle \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_3} \right\rangle^2} =$$

$$= \frac{1}{\|\frac{\partial}{\partial x_1}\|^2 \|\frac{\partial}{\partial x_3}\|^2} \left\langle \nabla_{\frac{\partial}{\partial x_1}} \nabla_{\frac{\partial}{\partial x_3}} \frac{\partial}{\partial x_3} - \nabla_{\frac{\partial}{\partial x_3}} \nabla_{\frac{\partial}{\partial x_1}} \frac{\partial}{\partial x_3} - \nabla_{\left[\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_3}\right]} \frac{\partial}{\partial x_3}, \frac{\partial}{\partial x_1} \right\rangle =$$

$$= x_3^2 x_3^2 \left\langle -\nabla_{\frac{\partial}{\partial x_1}} \frac{1}{x_3} \frac{\partial}{\partial x_3} + \nabla_{\frac{\partial}{\partial x_3}} \frac{1}{x_3} \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_1} \right\rangle =$$

$$= x_3^4 \left\langle -\frac{1}{x_3} \nabla_{\frac{\partial}{\partial x_1}} \frac{\partial}{\partial x_3} + \frac{\partial}{\partial x_3} \frac{1}{x_3} \frac{\partial}{\partial x_1} + \frac{1}{x_3} \nabla_{\frac{\partial}{\partial x_3}} \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_1} \right\rangle =$$

$$= x_3^4 \left\langle \frac{1}{x_3^2} \frac{\partial}{\partial x_1} - \frac{1}{x_3^2} \frac{\partial}{\partial x_1} + - \frac{1}{x_3^2} \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_1} \right\rangle = -x_3^4 \frac{1}{x_3^2} \frac{1}{x_3^2} = -1$$

Computations of $K(\frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3})$ and $K(\frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_1})$ are similar.

Remark. In fact, the plane spanned by vectors $\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_3}$ is tangent to vertical hyperbolic plane $x_2 = c$, so the corresponding sectional curvature is exactly the curvature of hyperbolic plane which is equal to -1. Thus, we could avoid all the computations. The same holds for the plane spanned by vectors $\frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}$. The plane spanned by vectors $\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}$ is tangent to a Euclidean hemisphere $(x_1 - a)^2 + (x_2 - b)^2 + x_3^2 = R^2$ which is also a hyperbolic plane in $\mathbb{H}^3$.

(c) By computations similar to ones done in (b), we obtain that

$$\left\langle R(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2})\frac{\partial}{\partial x_3}, \frac{\partial}{\partial x_1} \right\rangle = \left\langle R(\frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_1})\frac{\partial}{\partial x_3}, \frac{\partial}{\partial x_2} \right\rangle = \left\langle R(\frac{\partial}{\partial x_3}, \frac{\partial}{\partial x_1})\frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3} \right\rangle = 0$$

Now we see that for all vectors $\{v_1, v_2, v_2, v_4\} \subset \{\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}\}$ we have an equality

$$\langle R(v_1, v_2)v_3, v_4 \rangle = -\left(\langle v_1, v_3 \rangle \langle v_2, v_4 \rangle - \langle v_1, v_4 \rangle \langle v_2, v_3 \rangle \right)$$

By linearity, the equality above holds for any quadruple of tangent vectors. According to Problem 2.1, this implies that sectional curvature is constant and equal -1.

(d) Exactly the same as (a).

(e) The tangent space in every point of N is spanned by vectors $\{\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}, \frac{\partial}{\partial x_n}\}$. The restriction of the metric $\tilde{g}$ to this plane gives 3-dimensional symmetric bilinear form coinciding with one for $\mathbb{H}^3$.

(f) This follows from (e).

(g) Identical to (c).

1.8. Horosphere in hyperbolic 3-space

Consider a *horosphere*

$$M = \{x \in \mathbb{H}^3 \,|\, x_1^2 + x_2^2 + (x_3 - 1)^2 = 1\}$$

in real hyperbolic 3-space with metric g induced from $\mathbb{H}^3$.

(a) Parametrize M using spherical coordinates, and compute the induced metric.

(b) Compute the Christoffel symbols of (M, g).

(c) Compute the curvature tensor of (M, g). More precisely, prove that the curvature tensor is identically zero.

Solution: (a) Parametrize the horosphere by (φ, ϑ), where

$$(x_1, x_2, x_3) = (\cos\varphi \sin\vartheta, \sin\varphi \sin\vartheta, 1 + \cos\vartheta)$$

Then the metric g_{ij} can be written as

$$g_{11} = \frac{\sin^2\vartheta}{(1+\cos\vartheta)^2},\ g_{22} = \frac{1}{(1+\cos\vartheta)^2},\ g_{12} = g_{21} = 0.$$

(b) Computation of Christoffel symbols (as in Problem 1.7) gives

$$\Gamma^2_{11} = -\sin\vartheta, \quad \Gamma^1_{12} = \Gamma^1_{21} = \frac{1}{\sin\vartheta}, \quad \Gamma^2_{22} = \frac{\sin\vartheta}{1+\cos\vartheta},$$

while all the remaining ones are zero.

(c) Using (b), we obtain

$$\nabla_{\frac{\partial}{\partial\varphi}} \frac{\partial}{\partial\varphi} = -\sin\vartheta \frac{\partial}{\partial\vartheta}, \quad \nabla_{\frac{\partial}{\partial\varphi}} \frac{\partial}{\partial\vartheta} = \nabla_{\frac{\partial}{\partial\vartheta}} \frac{\partial}{\partial\varphi} = \frac{1}{\sin\vartheta} \frac{\partial}{\partial\varphi}, \quad \nabla_{\frac{\partial}{\partial\vartheta}} \frac{\partial}{\partial\vartheta} = \frac{\sin\vartheta}{1+\cos\vartheta} \frac{\partial}{\partial\vartheta}$$

We need to compute $\langle R(\frac{\partial}{\partial\varphi}, \frac{\partial}{\partial\vartheta})\frac{\partial}{\partial\varphi}, \frac{\partial}{\partial\vartheta}\rangle$ as a unique (up to permutation of indices) non-zero component of the curvature tensor.

$$R(\frac{\partial}{\partial\varphi}, \frac{\partial}{\partial\vartheta})\frac{\partial}{\partial\varphi} = \nabla_{\frac{\partial}{\partial\vartheta}} \nabla_{\frac{\partial}{\partial\varphi}} \frac{\partial}{\partial\varphi} - \nabla_{\frac{\partial}{\partial\varphi}} \nabla_{\frac{\partial}{\partial\vartheta}} \frac{\partial}{\partial\varphi} + \nabla_{\left[\frac{\partial}{\partial\varphi}, \frac{\partial}{\partial\vartheta}\right]} \frac{\partial}{\partial\varphi}$$

Since the commutator of $\frac{\partial}{\partial\varphi}$ and $\frac{\partial}{\partial\vartheta}$ is zero, we obtain

$$\begin{aligned} R(\frac{\partial}{\partial\varphi}, \frac{\partial}{\partial\vartheta})\frac{\partial}{\partial\varphi} &= \nabla_{\frac{\partial}{\partial\vartheta}} \nabla_{\frac{\partial}{\partial\varphi}} \frac{\partial}{\partial\varphi} - \nabla_{\frac{\partial}{\partial\varphi}} \nabla_{\frac{\partial}{\partial\vartheta}} \frac{\partial}{\partial\varphi} = \\ &= \nabla_{\frac{\partial}{\partial\vartheta}} (-\sin\vartheta) \frac{\partial}{\partial\vartheta} - \nabla_{\frac{\partial}{\partial\varphi}} \frac{1}{\sin\vartheta} \frac{\partial}{\partial\varphi} = -\cos\vartheta \frac{\partial}{\partial\vartheta} - \sin\vartheta \nabla_{\frac{\partial}{\partial\vartheta}} \frac{\partial}{\partial\vartheta} - \frac{1}{\sin\vartheta} \nabla_{\frac{\partial}{\partial\varphi}} \frac{\partial}{\partial\varphi} = \\ &= -\cos\vartheta \frac{\partial}{\partial\vartheta} - \sin\vartheta \frac{\sin\vartheta}{1+\cos\vartheta} \frac{\partial}{\partial\vartheta} - \frac{1}{\sin\vartheta}(-\sin\vartheta) \frac{\partial}{\partial\vartheta} = (-\cos\vartheta - \frac{\sin^2\vartheta}{1+\cos\vartheta} + 1) \frac{\partial}{\partial\vartheta} = 0 \end{aligned}$$